On ordered Γ-semihypergroups Containing Two-sided Bases

Wichayaporn Jantanan, Chanon Budpan and Weeradon Loesna*

Department of Mathematics, Faculty of Science, Buriram Rajabhat University, Buriram, 31000, Thailand

* Corresponding author. E-mail address: wichayaporn.jan@bru.ac.th and weeradon.loesna@gmail.com

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Abstract

The main purpose of this paper is to study the concept of an ordered Γ-semihypergroup containing two-sided bases that are studied analogously to the concept of Γ-semigroup containing two-sided bases considered by Thawhat Changpas and Pisit Kummoon in 2018. We introduce the notion of an ordered Γ-semihypergroup containing two-sided bases and describe some property of an ordered Γ-semihypergroup containing two-sided bases.

Keywords: ordered Γ-semihypergroup, two-sided bases, Γ-hyperideal

Introduction

In 1986, M. K. Sen and N. K. Saha (Sen & Saha, 1986). defined the notion of Γ-semigroup as a generalization of a semigroup. Also in (Fabrici, 1975). I. Fabrici has introduced and studied the concept of two-sided bases of semigroups. The notion and result of two-sided bases of semigroups have been extended to Γ-semigroup containing two-sided bases by T. Changphas and P. Kummoon (Changphas & Kummoon, 2018). The purpose of this paper is to introduce the concept of an ordered Γ-semihypergroup containing two-sided bases which extends from the concept of Γ-semigroup containing two-sided bases.

Hyperstructures represent a natural extension of classical algebraic structures and they were introduced by a French mathematician F. Marty (Marty, 1934). Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

Let H be a non-empty subset. Then the map $\circ : H \times H \to P^*(H)$ is called a hyperoperation, where $P^*(H)$ is the family of non-empty subset of H. $(H,\circ)$ is called a semihypergroup if for every $x,y,z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$. If for every $x \in H$, $x \circ H = H = H \circ x$, then $(H,\circ)$ is called a hypergroup. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$.

Preliminaries

In this section, we give some definitions that will be used in this paper.

Definition 1. (Davvaz, Dehkordi & Heidari, 2010). Let H and Γ be two non-empty sets. H is called a Γ-semihypergroup if every $\gamma \in \Gamma$ is a hyperoperation on H, $x\gamma y \subseteq H$ for every $x,y \in H$, and for every

$\alpha, \beta \in \Gamma$ and $x, y, z \in H$ we have $x\alpha(y\beta z) = (x\alpha y)\beta z$. Let A and B be two non-empty subsets of H and $\gamma \in \Gamma$. We define $A\gamma B = \bigcup_{a \in A, b \in B} a\gamma b$ and $A\Gamma B = \bigcup_{\gamma \in \Gamma} A\gamma B$.

Definition 2. (Davvaz & Omidi, 2017). $(H, \Gamma, \leq)$ is called an ordered Γ-semihypergroup if (H, Γ) is a Γ-semihypergroup and $(H, \leq)$ is a partially ordered set such that for any $x, y, z \in H$ and $\gamma \in \Gamma$, $x \leq y$ implies $z\gamma x \leq z\gamma y$ and $x\gamma z \leq y\gamma z$.

Here, if A and B are two non-empty subsets of H, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

Definition 3. (Kondo & Lekkoksung, 2013). A nonempty subset A of an ordered Γ-semihypergroup $(H, \Gamma, \leq)$ is called a sub Γ-semihypergroup of H if $A\Gamma A \subseteq A$.

Definition 4. (Kondo & Lekkoksung, 2013). A nonemty subset A of an ordered Γ-semihypergroup $(H, \Gamma, \leq)$ is called a left (resp. right) Γ-hyperideal of H if $H\Gamma A \subseteq A$ (resp. $A\Gamma H \subseteq A$) and $a \in A, b \leq a$ for $b \in H$ implies $b \in A$. A is called a two-side Γ-hyperideal (or simply called a Γ-hyperideal) of H if A is both a left and a right hyperideal of H.

Definition 5. (Davvaz & Omidi, 2017). Let K be a non-empty subset of an ordered Γ-semihypergroup $(H, \Gamma, \leq)$. We define $(K] := \{x \in H \mid x \leq k$ for some $k \in K\}$. For $K = \{k\}$, we write $(k]$ instead of $(\{k\}]$. If A and B are non-empty subsets of H, then we have

(1) $A \subseteq (A]$;

(2) $((A]] = (A]$;

(3) If $A \subseteq B$, then $(A] \subseteq (B]$;

(4) $(A]\Gamma(B] \subseteq (A\Gamma B]$;

(5) $((A]\Gamma(B]] = (A\Gamma B]$.

Definition 6. (Davvaz & Omidi, 2018). A proper Γ-hyperideal M of an ordered Γ-semihypergroup $(H, \Gamma, \leq)$ $(M \neq H)$ is said to be maximal if for any Γ-hyperideal A of $H, M \subseteq A \subseteq H$ implies $M = A$ or $A = H$.

Proposition 7. Let $(H, \Gamma, \leq)$ be an ordered Γ-semihypergroup and B_i be a Γ-hyperideal of H for each $i \in I$. If $\bigcap_{i \in I} B_i \neq \varnothing$ then $\bigcap_{i \in I} B_i$ is a Γ-hyperideal of H.

Proof. Assume that $\bigcap_{i \in I} B_i \neq \varnothing$. Suppose that $A = \bigcap_{i \in I} B_i \neq \varnothing$. We will show that $\bigcap_{i \in I} Bi$ is a Γ-hyperideal of H. First ,we let $a \in H\Gamma A$. We have $a \in h\gamma b_1$ for some $h \in H, \gamma \in \Gamma$ and $b_1 \in A$. Since $b_1 \in A = \bigcap_{i \in I} B_i$, so we obtain $b_1 \in B_i$. For any $i \in I, B_i$ is a Γ-hyperideal. Hence $a \in h\gamma b_1 \subseteq H\Gamma B_i \subseteq B_i$ for all $i \in I$. Thus $a \in \bigcap_{i \in I} B_i = A$. Therefore $H\Gamma A \subseteq A$. Next, we let $a \in A\Gamma H$. We have $a \in b_2\gamma h$ for some $b_2 \in A, \gamma \in \Gamma$ and $h \in H$. Since $b_2 \in A = \bigcap_{i \in I} B_i$, so we obtain $b_2 \in B_i$. For any $i \in I, B_i$ is a Γ-hyperideal. Hence $a \in b_2\gamma h \subseteq B_i\Gamma H \subseteq B_i$ for all $i \in I$. Thus $a \in \bigcap_{i \in I} B_i = A$. Therefore $A\Gamma H \subseteq A$. Finally, we show that, if $a \in \bigcap_{i \in I} B_i$ and $c \in H$ such that $c \leq a$ then $c \in \bigcap_{i \in I} B_i$. Let $a \in \bigcap_{i \in I} B_i$ and $c \in H$ such that

$c \leq a$. Since $a \in \bigcap_{i \in I} B_i$ and B_i is a Γ-hyperideal of H for all $i \in I$, we have $c \in B_i$ for all $i \in I$. Thus $c \in \bigcap_{i \in I} B_i$ for all $i \in I$. Hence $A = \bigcap_{i \in I} B_i$ is a Γ-hyperideal of H.

(Davvaz & Omidi, 2017). Let A be a non-empty subset of an ordered Γ-semihypergroup $(H, \Gamma, \leq)$. We denote by $I(A)$ is the Γ-hyperideal of H generated by A and $I(A)$ can show in the form of $I(A) = (A \cup H\Gamma A \cup A\Gamma H \cup H\Gamma A\Gamma H]$.

In particular, for an element $a \in H$, we write $I(\{a\})$ by $I(a)$ which is called the principal Γ-hyperideal of H generated by a. Thus, $I(a) = (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$.

Note that for any $b \in H$, we have that $(H\Gamma b \cup b\Gamma H \cup H\Gamma b\Gamma H]$ is a Γ-hyperideal of H. (Davvaz & Omidi, 2017). Finally, if A and B are two Γ-hyperideals of H then the union $A \cup B$ is a Γ-hyperideal of H.

Definition 8. Let $(H, \Gamma, \leq)$ be an ordered Γ-semihypergroup. A non-empty subset A of H is called a two-sided base of H if it satisfies the following two conditions:

(i) $H = (A \cup H\Gamma A \cup A\Gamma H \cup H\Gamma A\Gamma H]$;

(ii) if B is a subset of A such that $H = (B \cup H\Gamma B \cup B\Gamma H \cup H\Gamma B\Gamma H]$, then $B = A$.

Example 9. (Davvaz & Omidi, 2017). Let $H = \{a, b, c, d\}$ and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperoperations defined as follows.

γ	a	b	c	d
a	a	$\{a,b\}$	$\{c,d\}$	d
b	$\{a,b\}$	b	$\{c,d\}$	d
c	$\{c,d\}$	$\{c,d\}$	c	d
d	d	d	d	d

β	a	b	c	d
a	a	$\{a,b\}$	$\{c,d\}$	d
b	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	d
c	$\{c,d\}$	$\{c,d\}$	c	d
d	d	d	d	d

$$\leq := \{(a,a),(a,b),(b,b),(c,b),(c,c),(c,d),(d,b),(d,d)\}.$$

In (Davvaz & Omidi, 2017). $(H, \Gamma, \leq)$ is an ordered Γ-semihypergroups. Consider $A_1 = \{a\}$ and $A_2 = \{b\}$, we have A_1 and A_2 are two-sided bases of H. But $A_3 = \{a, b\}$ is not a two-sided base.

Example 10. (Davvaz & Omidi, 2018). Let $H = \{e, a, b, c, d\}$ and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperoperations defined as follows.

γ	e	a	b	c	d
e	e	e	e	e	e
a	e	$\{a,b\}$	b	b	b
b	e	b	b	b	b
c	e	c	c	c	c
d	e	d	d	d	d

β	e	a	b	c	d
e	e	e	e	e	e
a	e	a	a	a	a
b	e	a	$\{a,b\}$	a	a
c	e	c	c	c	c
d	e	d	d	d	d

$$\leq := \{(a,a),(a,b),(b,b),(c,c),(c,d),(d,d),(e,e)\}.$$

In (Davvaz & Omidi, 2018). $(H,\Gamma,\leq)$ is an ordered Γ-semihypergroups. Consider $A=\{e,b,d\}$ and $B=\{a,b,d\}$, we have A and B are two-sided bases of H. But $C=\{a\}$ is not a two-sided base.

In Example 9. and Example 10., it is observed that two-sided bases of H have same cardinality. This leads a proof in Theorem 4.

Definition 11. Let $(H,\Gamma,\leq)$ be an ordered Γ-semihypergroup. We define a **quasi-ordering** on H by for any $a,b\in H$,

$$a \preceq_I b \Leftrightarrow I(a)\subseteq I(b).$$

We write $a \prec_I b$ if $a \preceq_I b$ but $a\neq b$. It is clear that, for any a,b in H, $a\leq b$ implies $a \preceq_I b$.

Lamma 12. Let A be a two-sided base of an ordered Γ-semihypergroup $(H,\Gamma,\leq)$, and $a,b\in A$. If $a\in(H\Gamma b\cup b\Gamma H\cup H\Gamma b\Gamma H]$, then $a=b$.

Proof. Assume that $a\in(H\Gamma b\cup b\Gamma H\cup H\Gamma b\Gamma H]$, and suppose that $a\neq b$. Let $B=A\setminus\{a\}$. Since $a\neq b, b\in B$. To show that $I(A)\subseteq I(B)$, it suffices to show $A\subseteq I(B)$. Let $x\in A$. If $x\neq a$, then $x\in B$ and so $x\in I(B)$. If $x=a$, then by assumption we have $x=a\in(H\Gamma b\cup b\Gamma H\cup H\Gamma b\Gamma H]\subseteq I(b)\subseteq I(B)$. Thus $H=I(A)\subseteq I(B)\subseteq H$. This is contradiction. Hence $a=b$.

Main Results

In this part the algebraic structure of an ordered Γ-semihypergroup containing two-sided bases will be presented.

Theorem 1. A non-empty subset A of an ordered Γ-semihypergroup $(H,\Gamma,\leq)$ is a two-sided base of H if and only if A satisfies the following two conditions:

(i) For any $x\in H$ there exists $a\in A$ such that $x \preceq_I a$;

(ii) For any $a,b\in A$, if $a\neq b$, then neither $a \preceq_I b$ nor $b \preceq_I a$.

Proof. Assume first that A is a two-sided base of H. Then $I(A)=H$. Let $x\in H$, then $x\in(A\cup H\Gamma A\cup A\Gamma H\cup H\Gamma A\Gamma H]$. Since $x\in(A\cup H\Gamma A\cup A\Gamma H\cup H\Gamma A\Gamma H]$, we have $x\preceq y$ for some $y\in A\cup H\Gamma A\cup A\Gamma H\cup H\Gamma A\Gamma H$. There are four cases to consider :

Case 1: $y\in A$. Since $x\preceq y$, then we have $x \preceq_I y$, where $y\in A$.

Case 2: $y\in H\Gamma A$. Then $y\in h\gamma a$ for some $h\in H$, $\gamma\in\Gamma$ and $a\in A$.

By $y\in h\gamma a\subseteq H\Gamma a\subseteq(a\cup H\Gamma a\cup a\Gamma H\cup H\Gamma a\Gamma H]$, $H\Gamma y\subseteq H\Gamma(H\Gamma a)=(H\Gamma H)\Gamma a\subseteq H\Gamma a\subseteq(a\cup H\Gamma a\cup a\Gamma H\cup H\Gamma a\Gamma H]$, $y\Gamma H\subseteq(H\Gamma a)\Gamma H\subseteq(a\cup H\Gamma a\cup a\Gamma H\cup H\Gamma a\Gamma H]$ and $H\Gamma y\Gamma H\subseteq H\Gamma(H\Gamma a)\Gamma H\subseteq(a\cup H\Gamma a\cup a\Gamma H\cup H\Gamma a\Gamma H]$. Then $y\cup H\Gamma y\cup y\Gamma H\cup H\Gamma y\Gamma H\subseteq(a\cup H\Gamma a\cup a\Gamma H\cup H\Gamma a\Gamma H]$, so $(y\cup H\Gamma y\cup y\Gamma H\cup H\Gamma y\Gamma H]\subseteq(a\cup H\Gamma a\cup a\Gamma H\cup H\Gamma a\Gamma H]$. Thus $I(y)\subseteq I(a)$, *i.e.*, $y \preceq_I a$.

Case 3: $y\in A\Gamma H$. Then $y\in a\gamma h$ for some $h\in H$, $\gamma\in\Gamma$ and $a\in A$.

By $y \in a\gamma h \subseteq a\Gamma H \subseteq (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$, $H\Gamma y \subseteq H\Gamma(a\Gamma H) \subseteq (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$, $y\Gamma H \subseteq (a\Gamma H)\Gamma H \subseteq a\Gamma H \subseteq (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$ and $H\Gamma y\Gamma H \subseteq H\Gamma(H\Gamma a)\Gamma H \subseteq (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$. Then $y \cup H\Gamma y \cup y\Gamma H \cup H\Gamma y\Gamma H \subseteq (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$. Hence $(y \cup H\Gamma y \cup y\Gamma H \cup H\Gamma y\Gamma H] \subseteq (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$. Thus $I(y) \subseteq I(a)$, *i.e.*, $y \preceq_I a$.

Case 4: $y \in H\Gamma A\Gamma H$. Then $y \in h\gamma a_1\beta h_2$ for some $h_1, h_2 \in H$, $\gamma, \beta \in \Gamma$ and $a \in A$. By $y \in h\gamma a_1\beta h_2 \subseteq H\Gamma a\Gamma H \subseteq (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$, $H\Gamma y \subseteq H\Gamma(H\Gamma a\Gamma H) \subseteq H\Gamma a\Gamma H \subseteq (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$, $y\Gamma H \subseteq (H\Gamma a\Gamma H)\Gamma H \subseteq H\Gamma a\Gamma H \subseteq (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$ and $H\Gamma y\Gamma H \subseteq H\Gamma\ (H\Gamma a\Gamma H)\ \Gamma H \subseteq (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$. Then $y \cup H\Gamma y \cup y\Gamma H \cup H\Gamma y\Gamma H \subseteq (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$, so $(y \cup H\Gamma y \cup y\Gamma H \cup H\Gamma y\Gamma H] \subseteq (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$. Thus $I(y) \subseteq I(a)$, *i.e.*, $y \preceq_I a$. Hence condition (i) is true. Let a, b be elements of A such that $a \neq b$. Suppose $a \preceq_I b$. We set $B = A \setminus \{a\}$. Then $b \in B$. Let x be element of H. By (i), there exists c in A such that $x \preceq_I c$. There are two cases to consider. If $c \neq a$, then $c \in B$, thus $I(x) \subseteq I(c) \subseteq I(B)$. Hence $H = I(B)$. This is a contradition. If $c = a$, then $x \preceq_I b$, hence $x \in I(B)$ since $b \in B$. We have $H = I(B)$. This is a contradition. The case $b \preceq_I a$ is proved similarly. Thus (ii) true.

Conversely, assume that the conditions (i) and (ii) hold. We will show that A is a two-sided base of H. To show that $H = I(A)$. Let $x \in H$. By (i) , there exists $a \in A$ such that $I(x) \subseteq I(a)$. Then $x \in I(x) \subseteq I(a) \subseteq I(A)$. Thus $H \subseteq I(A)$ and $H = I(A)$. It remains to show that A is a minimal subset of H with the property: $H = I(A)$. Suppose that $H = I(B)$ for some $B \subset A$. Since $B \subset A$, there exists $a \in A$ and $a \notin B$. Next we show that $a \notin (B]$. If $a \in (B]$, then $a \leq y$ for some $y \in B$. So we have $a \preceq_I y$. This is a contradiction. Thus $a \notin (B]$. Since $a \in A \subseteq H = I(B)$ and $a \notin (B]$, it follows that $a \in (H\Gamma B \cup B\Gamma H \cup H\Gamma B\Gamma H]$. Since $a \in (H\Gamma B \cup B\Gamma H \cup H\Gamma B\Gamma H]$, we have $a \leq y$ for some $y \in H\Gamma B \cup B\Gamma H \cup H\Gamma B\Gamma H$. There are three cases to consider:

Case 1: $y \in H\Gamma B$. Then $y \in h\gamma b_1$ for some $b_1 \in B, \gamma \in \Gamma$ and $h \in H$. Since $a \leq y$ and $y \in b_1 \cup H\Gamma b_1 \cup b_1\Gamma H \cup H\Gamma b_1\Gamma H$, so $a \in (b_1 \cup H\Gamma b_1 \cup b_1\Gamma H \cup H\Gamma b_1\Gamma H]$. It follows that $I(a) \subseteq I(b_1)$. Hence, $a \preceq_I b_1$. This is a contradiction.

Case 2: $y \in B\Gamma H$. Then $y \in b_2\gamma h$ for some $b_2 \in B, \gamma \in \Gamma$ and $h \in H$. Since $a \leq y$ and $y \in b_2 \cup H\Gamma b_2 \cup b_2\Gamma H \cup H\Gamma b_2\Gamma H$, so $a \in (b_2 \cup H\Gamma b_2 \cup b_2\Gamma H \cup H\Gamma b_2\Gamma H]$. It follows that $I(a) \subseteq I(b_2)$. Hence, $a \preceq_I b_2$. This is a contradiction.

Case 3: $y \in H\Gamma B\Gamma H$. Then $y \in h_1\gamma_1 b_3\gamma_2 h_2$ for some $b_3 \in B, \gamma_1, \gamma_2 \in \Gamma$ and $h_1, h_2 \in H$. Since $a \leq y$ and $y \in b_3 \cup H\Gamma b_3 \cup b_3\Gamma H \cup H\Gamma b_3\Gamma H$, so $a \in (b_3 \cup H\Gamma b_3 \cup b_3\Gamma H \cup H\Gamma b_3\Gamma H]$. Thus $I(a) \subseteq I(b_3)$. Hence $a \preceq_I b_3$. This is a contradiction.

Therefore, A is a two-sided base of H as required, and the proof is completed.

Theorem 2. Let A be a two-sided base of an ordered Γ-semihypergroup $(H, \Gamma, \leq)$ such that $I(a) = I(b)$ for some a in A and b in H. If $a \neq b$, then H contains at least two two-sided base.

Proof. Assume that $a \neq b$. Suppose that $b \in A$. Since $a \preceq_I b$ and $a \neq b$, it follows that $a \in (H\Gamma b \cup b\Gamma H \cup H\Gamma b\Gamma H]$. By Lemma 12., we obtain $a = b$. This is a contradiction. Thus $b \in H \setminus A$. Let $B := (A \setminus \{a\}) \cup \{b\}$. Since $b \in B$, we have $b \notin A$, and $B \not\subseteq A$. Hence $A \neq B$. We will show that B

is a two-sided base of H. To show that B satisfies (i) in Theorem 1., let $x \in H$. Since A is a two-sided base of H, there exists $c \in A$ such that $x \preceq_I c$. If $c \neq a$, then $c \in B$. If $c = a$, then $x \preceq_I a$. Since $a \preceq_I b, x \preceq_I a \preceq_I b$. Then $x \preceq_I b$. To show that B satisfies (ii) in Theorem 1., let $c_1, c_2 \in B$ be such that $c_1 \neq c_2$. We will show that neither $c_1 \preceq_I c_2$ nor $c_2 \preceq_I c_1$. Since $c_1 \in B$ and $c_2 \in B$, we have $c_1 \in A \setminus \{a\}$ or $c_1 = b$ and $c_2 \in A \setminus \{a\}$ or $c_2 = b$. There are four cases to consider:

Case 1: $c_1 \in A \setminus \{a\}$ and $c_2 \in A \setminus \{a\}$. By Theorem 1. (ii), this implies neither $c_1 \preceq_I c_2$ nor $c_2 \preceq_I c_1$.

Case 2: $c_1 \in A \setminus \{a\}$ and $c_2 = b$. If $c_1 \preceq_I c_2$, then $c_1 \preceq_I b$. Since $b \preceq_I a, c_1 \preceq_I b \preceq_I a$. Thus $c_1 \preceq_I a$ where $a, c_1 \in A$. By Theorem 1. (ii), $c_1 = a$. This is a contradiction. If $c_2 \preceq_I c_1$, then $b \preceq_I c_1$. Since $a \preceq_I b, a \preceq_I b \preceq_I c_1$. So $a \preceq_I c_1$ where $a, c_1 \in A$. By Theorem 1. (ii), $a = c_1$. This is a contradiction.

Case 3: $c_2 \in A \setminus \{a\}$ and $c_1 = b$. If $c_1 \preceq_I c_2$, then $b \preceq_I c_2$. Since $a \preceq_I b, a \preceq_I b \preceq_I c_2$. Hence $a \preceq_I c_2$ where $a, c_2 \in A$. By Theorem 1. (ii), $a = c_2$. This is a contradiction. If $c_2 \preceq_I c_1$, then $c_2 \preceq_I b$. Since $b \preceq_I a, c_2 \preceq_I b \preceq_I a$. Thus $c_2 \preceq_I a$ where $a, c_2 \in A$. By Theorem 1. (ii), $c_2 = a$. This is a contradiction.

Case 4: $c_1 = b$ and $c_2 = b$. This is impossible.

Thus B satisfies (i) and (ii) in Theorem 1. Therefore, B is a two-sided base of H.

Corollary 3. Let A be a two-sided base of an ordered Γ-semihypergroup $(H, \Gamma, \leq)$, and let $a \in A$. If $I(x) = I(a)$ for some $x \in H, x \neq a$, then x belongs to two-sided base of H, which is different from A.

Theorem 4. Let A and B be any two-sided bases of an ordered Γ-semihypergroup $(H, \Gamma, \leq)$. Then A and B have the same cardinality.

Proof. Let $a \in A$. Since B is a two-sided base of H and $a \in H$, by Theorem 1.(i) there exists an element $b \in B$ such that $a \preceq_I b$. Since A is a two-sided base of H, by Theorem 1.(i) there exists $a^* \in A$ such that $b \preceq_I a^*$. So $a \preceq_I b \preceq_I a^*$, i.e., $a \preceq_I a^*$. By Theorem 1.(ii), $a = a^*$. Hence $I(a) = I(b)$.

Define a mapping $\varphi : A \to B$ by $\varphi(a) = b$ for all $a \in A$.

To show that φ is well-defined, let $a_1, a_2 \in A$ be such that $a_1 = a_2, \varphi(a_1) = b_1$ and $\varphi(a_2) = b_2$ for some $b_1, b_2 \in B$. Then $I(a_1) = I(b_1)$ and $I(a_2) = I(b_2)$. Since $a_1 = a_2$, $I(a_1) = I(a_2)$. Hence $I(a_1) = I(a_2) = I(b_1) = I(b_2)$, i. e., $b_1 \preceq_I b_2$ and $b_2 \preceq_I b_1$. By Theorem 1. (ii), $b_1 = b_2$. Thus $\varphi(a_1) = \varphi(a_2)$. Therefore, φ is well-defined. We will show that φ is one-one. Let $a_1, a_2 \in A$ be such that $\varphi(a_1) = \varphi(a_2)$. Since $\varphi(a_1) = \varphi(a_2), \varphi(a_1) = \varphi(a_2) = b$ for some $b \in B$. So $I(a_2) = I(a_1) = I(b)$. Since $I(a_2) = I(a_1), a_1 \preceq_I a_2$ and $a_2 \preceq_I a_1$. This implies $a_1 = a_2$. Therefore φ is one-one. We will show that φ is onto. Let $b \in B$. Since A is a two-sided base of H, by Theorem 1.(i) there exists an element $a \in A$ such that $b \preceq_I a$. Since B is a two-sided base of H, by Theorem 1.(i) there exists an element $b^* \in B$ such that $a \preceq_I b^*$. So $b \preceq_I a \preceq_I b^*$, i.e., $b \preceq_I b^*$. By Theorem 1.(ii), $b = b^*$. Hence $I(a) = I(b)$. Thus $\varphi(a) = b$. Therefore, φ is onto. This completes the proof.

If a two-sided base A of an ordered Γ-semihypergroup $(H, \Gamma, \leq)$ is a two-sided Γ-hyperideal of H, then $H = (A \cup H\Gamma A \cup A\Gamma H \cup H\Gamma A\Gamma H] \subseteq (A] = A$. Hence $H = A$. The converse statement is obvious. Then we conclude that.

Remark 5. It is observed that a two-sided base A of an ordered Γ-semihypergroup $(H, \Gamma, \leq)$ is a two-sided Γ-hyperideal of H if and only if $A = H$.

Theorem 6. Let A be a two-sided base of an ordered Γ-semihypergroup $(H,\Gamma,\leq)$. If A is a sub Γ-semihypergroup of H then $A=\{a\}$ with $a\in a\gamma a$ for all $\gamma\in\Gamma$.

Proof. Assume that A is a sub Γ-semihypergroup H. Let $a,b\in A$ and $\gamma\in\Gamma$. Since A is a sub Γ-semihyperigroup of H, $a\gamma b\subseteq A$. Setting $c\in a\gamma b$; thus $c\in H\Gamma b\subseteq H\Gamma b\cup b\Gamma H\cup H\Gamma b\Gamma H\subseteq (H\Gamma b\cup b\Gamma H\cup H\Gamma b\Gamma H]$. By Lemma 12, $c=b$. Similarly, $c\in a\Gamma H\subseteq H\Gamma a\cup a\Gamma H\cup H\Gamma a\Gamma H\subseteq (H\Gamma a\cup a\Gamma H \cup H\Gamma a\Gamma H]$ By Lemma 12., $c=a$. We have $a=b$. Therefore, $A=\{a\}$ with $a\in a\gamma a$ for all $a\in A$ and $\gamma\in\Gamma$.

In Example 9., we have $A_2=\{b\}$ is a two-sided base of an ordered Γ-semihypergroup H, such that $b\in b\gamma b$ for all $\gamma\in\Gamma$. But $A_2=\{b\}$ is not a sub Γ-semihypergroup of H. This shows that the converse statement is not valid in general.

Theorem 7. Let $(H,\Gamma,\leq)$ be an ordered Γ-semihypergroup and let T be an union of all two-sided bases of H. Then $H\setminus T$ is either empty set or a Γ-hyperideal of H.

Proof. Assume that $H\setminus T\neq\varnothing$. We will show that $H\setminus T$ is a Γ-hyperideal of H. Let $a\in H\setminus T$, $x\in H$ and $\gamma\in\Gamma$. To show that $x\gamma a\subseteq H\setminus T$ and $a\gamma x\subseteq H\setminus T$, we suppose that $x\gamma a\not\subseteq H\setminus T$. Then there exists $b\in x\gamma a$ such that $b\in T$. Hence $b\in A$ for some a two-sided base A of H. Then $b\in H\Gamma a$. By $b\in H\Gamma a\subseteq a\cup a\Gamma H\cup H\Gamma a\cup H\Gamma a\Gamma H\subseteq (a\cup a\Gamma H\cup H\Gamma a\cup H\Gamma a\Gamma H]=I(a)$, so $I(b)\subseteq I(a)$. Next, we will show that $I(b)\subset I(a)$. Suppose that $I(b)=I(a)$. Since $a\in H\setminus T$ and $b\in A, a\neq b$. Since $I(b)=I(a)$ and Corollary 3., we conclude that $a\in T$. This is a contradiction. Thus $I(b)\subset I(a)$, i.e., $b\prec_I a$. Since A is a two-sided base of H and $a\in H\setminus T$, by Theorem 1.(i) there exists $b_1\in A$ such that $a\preceq_I b_1$. Since $b\prec_I a\preceq_I b_1, b\preceq_I b_1$. This contradicts to the condition (ii) of Theorem 1. Thus $x\gamma a\subseteq H\setminus T$. Similarly, we can show that $a\gamma x\subseteq H\setminus T$. Let $x\in H\setminus T$, $y\in H$ such that $y\leq x$. We will show that $y\in H\setminus T$. Suppose that $y\in T$, then $y\in A$ for some a two-sided base A of H. Since A is a two-sided base of H, by Theorem 1.(i) there exists an element $z\in A$ such that $x\preceq_I z$. Since $y\preceq_I x$ and $x\preceq_I z$, we have $y\preceq_I z$. This is a contradiction. Therefore $y\notin T$ then $y\in H\setminus T$. Hence $H\setminus T$ is a Γ-hyperideal of H.

Theorem 8. Let $(H,\Gamma,\leq)$ be an ordered Γ-semihypergroup and $\varnothing\neq T\subset H$. If H contains a proper Γ-hyperideal of H containing every proper Γ-hyperideal of H, denoted by M^*, then the following statements are equivalent:

(i) $H\setminus T$ is a maximal proper Γ-hyperideal of H.

(ii) For every element $a\in T, T\subseteq I(a)$;

(iii) $H\setminus T=M^*$;

(iv) Every two-sided base of H is a one-element base.

Proof. (i)$\Leftrightarrow$(ii). Assume that $H\setminus T$ is a maximal proper Γ-hyperideal of H. Let $a\in T$. Suppose that $T\not\subseteq I(a)$. Since $T\not\subseteq I(a)$, there exists $x\in T$ such that $x\notin I(a)$. So $x\notin H\setminus T$. Since $x\notin I(a)$, $x\notin H\setminus T$ and $x\in H$, we have $(H\setminus T)\cup I(a)\subset H$. Thus $(H\setminus T)\cup I(a)$ is a proper Γ-hyperideal of H. Hence $H\setminus T\subset (H\setminus T)\cup I(a)$. This contradicts to the maximality of $H\setminus T$.

Conversely, assume that for every element $a\in T, T\subseteq I(a)$. We will show that $H\setminus T$ is a maximal proper Γ-hyperideal of H. Since $a\in T$, $a\notin H\setminus T$. So $H\setminus T\subset H$. Since $T\subset H, H\setminus T\neq\varnothing$. By Theorem 7., $H\setminus T$ is a proper Γ-hyperideal of H. Suppose that M is a proper Γ-hyperideal of H such that $H \setminus T \subset M \subset H$. Since $H \setminus T \subset M$, there exists $x \in M$ such that $x \notin H \setminus T$, i.e., $x \in T$. Then $x \in M \cap T$. So $M \cap T \neq \varnothing$. Let $c \in M \cap T$. Then $c \in M$ and $c \in T$. Since $c \in M, H\Gamma c \subseteq H\Gamma M \subseteq M$ $, c\Gamma H \subseteq M\Gamma H \subseteq M$ and $H\Gamma c\Gamma H \subseteq H\Gamma M\Gamma H \subseteq M$. Then $I(c) = (c \cup H\Gamma c \cup c\Gamma H \cup H\Gamma c\Gamma H] \subseteq M$. Since $c \in T$, by assumption we have $T \subseteq I(c)$. Hence $H = (H \setminus T) \cup T \subseteq (H \setminus T) \cup I(c) \subseteq M \subset H$. Thus $M = H$. This is a contradiction. Therefore $H \setminus T$ is a maximal proper Γ-hyperideal of H.

(iii) $\Leftrightarrow$ (iv). Assume that $H \setminus T = M^*$. Since $H \setminus T = M^*$, $H \setminus T$ is a maximal proper Γ-hyperideal of H. By (i) $\Leftrightarrow$ (ii), for every $a \in T$, $T \subseteq I(a)$. First, we will show that for every $a \in T, H \setminus T \subseteq I(a)$. Suppose that $H \setminus T \not\subseteq I(a)$ for some $a \in T$. Then $I(a) \neq H$. Hence $I(a)$ is a proper Γ-hyperideal of H. Thus $I(a) \subseteq M^* = H \setminus T$. Then $I(a) \subseteq H \setminus T$. Since $a \in I(a), a \in H \setminus T$, i.e., $a \notin T$. This is a contradiction. Thus $H \setminus T \subseteq I(a)$ for every $a \in T$. Since $H \setminus T \subseteq I(a)$ and $T \subseteq I(a)$ for every $a \in T$, it follows that $H = (H \setminus T) \cup T \subseteq I(a) \cup I(a) = I(a) \subseteq H$. So $H = I(a)$ for every $a \in T$. Therefore $\{a\}$ is a two-sided base of H. Let A be a two-sided base of H. We will show that $a = b$ for all $a, b \in A$. Suppose that exist $a, b \in A$ such that $a \neq b$. Since A is a two-sided base of H, $A \subseteq T$. This is $a \in T$. So $H = I(a)$. Since $b \in H = I(a)$ and $b \neq a, b \in (H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H]$. By Lemma 12., $a = b$. This is a contradiction. Therefore, every two-sided base of H is a one element base.

Conversely, assume that every two-sided base of H is a one element base. Then $H = I(a)$ for all $a \in T$. We will show that $H \setminus T = M^*$. The statement that $H \setminus T$ is a maximal proper Γ-hyperideal of H follows from the proof (i) $\Leftrightarrow$ (ii). Let M be a Γ-hyperideal of H such that M is not contained in $H \setminus T$. Then $T \cap M \neq \varnothing$. Let $a \in T \cap M$. Hence $a \in T$ and $a \in M$. So $H\Gamma a \subseteq H\Gamma M \subseteq M, a\Gamma H \subseteq M\Gamma H \subseteq M$ and $H\Gamma a\Gamma H \subseteq H\Gamma M\Gamma H$. So we have $I(a) = (a \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H] \subseteq M$. Hence $H = I(a) \subseteq M$ $\subseteq H$. Therefore $M = H$.

(i) $\Leftrightarrow$ (iii). Assume that $H \setminus T$ is a maximal proper Γ-hyperideal of H. Next, we will show that $H \setminus T = M^*$. Since $H \setminus T$ is a proper Γ-hyperideal of H, $H \setminus T \subseteq M^* \subset H$. By assumption, $H = M^*$ or $H \setminus T = M^*$. Hence $H \setminus T = M^*$. The converse statement is obvious.

Conclusion and Discussion

In this paper, we prove that a non-empty subset A of an ordered Γ-semihypergroup $(H, \Gamma, \leq)$ is a two-sided base of H if and only if A satisfies the following two conditions: (i) For any $x \in H$ there exists $a \in A$ such that $x \preceq_I a$; (ii) For any $a, b \in A$, if $a \neq b$, then neither $a \preceq_I b$ nor $b \preceq_I a$. Also we prove that if A and B be any two-sided bases of an ordered Γ-semihypergroup $(H, \Gamma, \leq)$, then A and B have same cardinality. Finally, let $(H, \Gamma, \leq)$ be an ordered Γ-semihypergroup and let T be an union of all two-sided bases of H we prove that $H \setminus T$ is either empty set or a Γ-hyperideal of H.

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