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ON ORDERED LA-Γ-SEMIGROUPS CONTAINING TWO-SIDED BASES

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Abstract. *Two-sided base is the smallest set generated two-sided ideal under some condition. The aim of this paper is to introduce the concept of two-sided bases of an ordered LA-Γ-semigroup with left identity. We give a characterization when a non-empty subset of an ordered LA-Γ-semigroup with left identity is a two-sided base of an ordered Γ-semigroup with left identity. Finally, a characterization when the complement of the union of all two-sided bases of an ordered Γ-semigroup with left identity is maximal will be given.*

Keywords: Ordered LA-Γ-semigroups, Γ-ideals, Two-sided bases, Maximal proper Γ-ideals

1. **Introduction.** Based on the notion of two-sided ideals of a semigroup generated by a non-empty set, the concept of two-sided bases of a semigroup has been introduced and studied by Fabrici [1]. Later, Changpas and Kummoon [2] studied and described the structure of a Γ-semigroup containing two-sided bases. The structure of a Γ-semigroup was introduced by Sen [3] as a generalization of ternary semigroup and semigroup and the structure of an LA-semigroup was introduced by Kazim and Naseeruddin [4] as a generalization of commutative semigroups. The structure of an LA-Γ-semigroups (Γ-AG-groupoid), where Γ is a non-empty set, was given by Shah and Rehman [5]. The concept of an ordered LA-Γ-semigroups was introduced by Khan et al. [6]. This algebraic structure is a generalization of LA-Γ-semigroups, also see [7, 8]. The purpose of this paper is to introduce the concept of two-sided bases of an ordered LA-Γ-semigroup, and extend results in [1] to ordered LA-Γ-semigroups. In Section 2, we recall some basic definitions and results of ordered LA-Γ-semigroups. In Section 3, we define two-sided bases of ordered LA-Γ-semigroups and give their basic results. Section 4 is the main part of this paper, and we show remarkable results of two-sided bases of ordered LA-Γ-semigroups. Finally, Section 5 concludes the paper.

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2. **Ordered LA-Γ-Semigroups.** We provide some definitions and results which will be used for this paper.

Definition 2.1. ([5]) *Let S and Γ be non-empty sets, then S is called an* LA-Γ-semigroup *if there exists a mapping $S \times \Gamma \times S \to S$ written as (a, γ, b) and denoted by $a\gamma b$ such that S satisfied the left invertive law $(a\gamma b)\beta c = (c\gamma b)\beta a$ for all $a, b, c \in S$ and $\gamma, \beta \in \Gamma$.*

Definition 2.2. ([5]) *An element e of an LA-Γ-semigroup S is called a* left identity *if $e\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$.*

Lemma 2.1. ([5]) *If S is an LA-Γ-semigroup with left identity e, then $S\Gamma S = S$ and $S = e\Gamma S = S\Gamma e$.*

Proposition 2.1. ([9]) *Let S be an LA-Γ-semigroup.*

(1) *Every LA-Γ-semigroup with left identity satisfies the equalities $a\gamma(b\beta c) = b\gamma(a\beta c)$ and $(a\gamma b)\beta(c\alpha d) = (d\gamma c)\beta(b\alpha a)$ for all $a, b, c, d \in S$ and $\gamma, \beta, \alpha \in \Gamma$.*
(2) *An LA-Γ-semigroup S is Γ-medial, i.e., $(a\gamma b)\beta(c\gamma d) = (a\gamma c)\beta(b\alpha d) = (a\gamma c)\beta(b\alpha d)$ for all $a, b, c, d \in S$ and $\gamma, \beta, \alpha \in \Gamma$.*

Definition 2.3. ([6]) *An* ordered LA-Γ-semigroup *S (abbreviated as a po-LA-Γ-semigroup) is a structure $(S, \Gamma, \cdot, \leq)$ in which the following conditions hold.*

(1) *$(S, \Gamma, \cdot)$ is an LA-Γ-semigroup.*
(2) *$(S, \leq)$ is a poset (i.e., reflexive, anti-symmetric and transitive).*
(3) *For all a, b and $x \in S$, $a \leq b$ implies $a\alpha x \leq b\alpha x$ and $x\alpha a \leq x\alpha b$ for all $\alpha \in \Gamma$.*

Throughout this paper, unless stated otherwise, S stands for an ordered LA-Γ-semigroup. For a non-empty subsets A, B of an ordered LA-Γ-semigroup S, we defined

$$A\Gamma B = \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\} \text{ and } (A] = \{t \in S \mid t \leq a, \text{ for some } a \in A\}.$$

In particular, we write $B\Gamma a$ instead for $B\Gamma\{a\}$, $a\Gamma B$ instead for $\{a\}\Gamma B$, $a \cup B\Gamma a \cup a\Gamma s \cup (S\Gamma a)\Gamma S$ instead for $\{a\} \cup B\Gamma a \cup a\Gamma s \cup (S\Gamma a)\Gamma S$ and $(a]$ instead for $(\{a\}]$.

Definition 2.4. [7] *A non-empty subset A of an ordered LA-Γ-semigroup S is called an* LA-Γ-subsemigroup *of S if $A\Gamma A \subseteq A$.*

Definition 2.5. [6] *A non-empty subset A of an ordered LA-Γ-semigroup S is called a* left (resp. right) Γ-ideal *of S if (i) $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$) and (ii) if $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$. A non-empty subset A of an ordered LA-Γ-semigroup S is called a* Γ-ideal *of S if is both a left and right Γ-ideal of S.*

Definition 2.6. *A proper Γ-ideal A of an ordered LA-Γ-semigroup $S(A \neq S)$ is said to be* maximal *if for any Γ-ideal B of S, $A \subseteq B \subseteq S$ implies $A = B$ or $B = S$.*

Lemma 2.2. ([6]) *Let S be an ordered LA-Γ-semigroup, and then the following statements are true.*

(1) *$A \subseteq (A]$, for all $A \subseteq S$.*
(2) *If $A \subseteq B \subseteq S$ then $(A] \subseteq (B]$.*
(3) *$(A]\Gamma(B] \subseteq (A\Gamma B]$, for all subsets A, B of S.*
(4) *$(A] = ((A]]$, for all $A \subseteq S$.*
(5) *For every left (resp. right) Γ-ideal T of S, $(T] = T$.*
(6) *$((A]\Gamma(B]] \subseteq (A\Gamma B]$, for all subsets A, B of S.*
(7) *$(A \cup B] = (A] \cup (B]$, for all subsets A, B of S.*
(8) *If A and B are two Γ-ideal of S, then the union $A \cup B$ is a Γ-ideal of S.*

Lemma 2.3. *Let S be an ordered LA-Γ-semigroup and A_i be a Γ-ideal of S for all $i \in I$. If $\bigcap_{i \in I} A_i \neq \varnothing$, then $\bigcap_{i \in I} A_i$ is a Γ-ideal of S.*

Proof: It is obvious. □

Let A be a non-empty subset of an ordered LA-Γ-semigroup S. The intersection of all Γ-ideals of S containing A is the smallest Γ-ideal of S generated by A and is denoted by $(A)_T$.

Lemma 2.4. *Let A be a non-empty subset of an ordered LA-Γ-semigroup S with left identity e. Then $(A)_T = (A \cup S\Gamma A \cup A\Gamma S \cup (S\Gamma A)\Gamma S]$.*

Proof: Straightforward. □

For an element $a \in S$, we write $(\{a\})_T$ by $(a)_T$ which is called the principal Γ-ideal of S generated by a. Thus, $(a)_T = (a \cup S\Gamma a \cup a\Gamma S \cup (S\Gamma a)\Gamma S]$.

Corollary 2.1. *Let S be an ordered LA-Γ-semigroup with left identity. Then $(S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]$ is a Γ-ideal of S for all $b \in S$.*

3. **Two-Sided Bases of Ordered LA-Γ-Semigroups.** We begin this section with the definition of two-sided bases of an ordered LA-Γ-semigroup with left identity as follows.

Definition 3.1. *Let S be an ordered LA-Γ-semigroup with left identity. A non-empty subset A of S is called a* two-sided base *of S if it satisfies the following two conditions.*

(1) $S = (A \cup S\Gamma A \cup A\Gamma S \cup (S\Gamma A)\Gamma S]$.

(2) *If B is a subset of A such that $S = (B \cup S\Gamma B \cup B\Gamma S \cup (S\Gamma B)\Gamma S]$, then $B = A$.*

Example 3.1. *Let $S = \{a, b, c, d, e\}$ and $\Gamma = \{\gamma\}$ with multiplication defined by*

γ	a	b	c	d	c
a	a	a	a	a	a
b	a	b	c	d	c
c	a	c	b	c	d
d	a	d	c	b	c
c	a	c	d	c	b

and $\leq = \{(a, a), (b, b), (c, c), (d, d), (c, c), (a, b), (a, c), (a, d), (a, c)\}$. Then S is an ordered LA-Γ-semigroup with left identity b. We have the two-sided bases of S are $A_1 = \{b\}$, $A_2 = \{c\}$, $A_3 = \{d\}$ and $A_4 = \{e\}$. However, $A_5 = \{a\}$ is not a two-sided base.

Example 3.2. *Let $S = \{a, b, c, d, e\}$ and $\Gamma = \{\alpha\}$ with multiplication defined by*

α	a	b	c	d	c
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	d	c	c
d	a	b	c	d	c
c	a	b	c	c	d

and $\leq = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b)\}$. Then S is an ordered LA-Γ-semigroup with left identity d. We have the two-sided bases of S are $A_1 = \{c\}$, $A_2 = \{d\}$ and $A_3 = \{e\}$. However, $A_4 = \{a\}$ and $A_5 = \{b\}$ are not a two-sided bases.

To characterize when a non-empty subset of ordered LA-Γ-semigroup S with left identity is a two-sided base of the ordered LA-Γ-semigroup S with left identity we need the quasi-ordering defined as follows.

Definition 3.2. *Let S be an ordered LA-Γ-semigroup. We define a* quasi-ordering *on S for any $a, b \in S$, $a \leq_I b \Leftrightarrow (a)_T \subseteq (b)_T$.*

We write $a <_I b$ if $a \leq_I b$ but $a \neq b$, i.e., $a_T \subset b_T$.

The following example shows that the order $\leq_I$ defined above is not, in general, a partial order.

Example 3.3. *From Example 3.2, we have that* $(c)_T \subseteq (d)_T$ *(i.e.,* $c \leq_I d$*) and* $(d)_T \subseteq (c)_T$ *(i.e.,* $d \leq_I c$*), but* $c \neq d$*. Thus,* $\leq_I$ *is not a partial order on* S*.*

Lemma 3.1. *Let* S *be an ordered LA-*Γ*-semigroup. For any* $a, b \in S$*, if* $a \leq b$*, then* $a \leq_I b$*.*

Proof: Let $a, b \in S$ such that $a \leq b$. We will show that $a \leq_I b$, i.e., $(a)_T \subseteq (b)_T$. Let $x \in (a)_T$. Since $x \in (a)_T = (a \cup S\Gamma a \cup a\Gamma S \cup (S\Gamma a)\Gamma S]$, $x \leq y$ for some $y \in a \cup S\Gamma a \cup a\Gamma S \cup (S\Gamma a)\Gamma S$. There are four cases to consider.

Case 1: $y = a$. Then $x \leq a \leq b$, so $x \leq b$ where $b \in b \cup S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S$. We have that $x \in (b \cup S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]$; thus, $x \in (b)_T$. So $a \in (b)_T$.

Case 2: $y \in S\Gamma a$. Then $y = s\gamma a$ for some $s \in S$, $\gamma \in \Gamma$. Since $a \leq b$, then $s\gamma a \leq s\gamma b$ and $s\gamma b \in S\Gamma b \subseteq b \cup S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S$. Since $x \leq y \leq s\gamma b$ where $s\gamma b \in b \cup S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S$, $x \in (b)_T$. So $S\Gamma a \subseteq (b)_T$.

Case 3: $y \in a\Gamma S$. Then $y = a\gamma s$ for some $s \in S$, $\gamma \in \Gamma$. Since $a \leq b$, then $a\gamma s \leq b\gamma s$ and $b\gamma s \in b\Gamma S \subseteq b \cup S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S$. Since $x \leq y \leq b\gamma s$ where $b\gamma s \in b \cup S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S$, $x \in (b)_T$. So $a\Gamma S \subseteq (b)_T$.

Case 4: $y \in (S\Gamma a)\Gamma S$. Then $y = (s_1\gamma a)\beta s_2$ for some $s_1, s_2 \in S$, $\gamma, \beta \in \Gamma$. Since $a \leq b$, then $s_1\gamma a \leq s_1\gamma b$ and $(s_1\gamma a)\beta s_2 \leq (s_1\gamma b)\beta s_2$ where $(s_1\gamma b)\beta s_2 \in (S\Gamma b)\Gamma S \subseteq b \cup S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S$. Since $x \leq y \leq (s_1\gamma b)\beta s_2$ where $(s_1\gamma b)\beta s_2 \in (S\Gamma b)\Gamma S \subseteq b \cup S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S$, $x \in (b)_T$. So $(S\Gamma a)\Gamma S \subseteq (b)_T$. Hence $a \cup S\Gamma a \cup a\Gamma S \cup (S\Gamma a)\Gamma S \subseteq (b)_T$ and so $(a)_T = (a \cup S\Gamma a \cup a\Gamma S \cup (S\Gamma a)\Gamma S] \subseteq ((b)_T] = (b)_T$. Therefore, $(a)_T \subseteq (b)_T$, i.e., $a \leq_I b$. □

Lemma 3.2. *Let* A *be a two-sided base of an ordered LA-*Γ*-semigroup* S *with left identity and let* $a, b \in A$*. If* $a \in (S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]$*, then* $a = b$*.*

Proof: Assume that $a, b \in A$ such that $a \in (S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]$, and suppose that $a \neq b$. Let $B = A \backslash \{a\}$. Since $a \neq b$, $b \in B$. To show that $(A)_T \subseteq (B)_T$, we let $x \in (A \cup S\Gamma A \cup A\Gamma S \cup (S\Gamma A)\Gamma S]$. Then $x \leq z$ for some $z \in A \cup S\Gamma A \cup A\Gamma S \cup (S\Gamma A)\Gamma S$. There are four cases to consider.

Case 1: $z \in A$. If $z \neq a$, then $z \in B \subseteq (B \cup S\Gamma B \cup B\Gamma S \cup (S\Gamma B)\Gamma S]$. Since $x \leq z$ and $z \in (B \cup S\Gamma B \cup B\Gamma S \cup (S\Gamma B)\Gamma S]$, $x \in ((B \cup S\Gamma B \cup B\Gamma S \cup (S\Gamma B)\Gamma S]] = (B \cup S\Gamma B \cup B\Gamma S \cup (S\Gamma B)\Gamma S]$. Thus, $x \in (B)_T$. If $z = a$, then by assumption we have $z = a \in (S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S] \subseteq (B \cup S\Gamma B \cup B\Gamma S \cup (S\Gamma B)\Gamma S]$. Since $x \leq z$ and $z \in (B \cup S\Gamma B \cup B\Gamma S \cup (S\Gamma B)\Gamma S]$, then $x \in ((B \cup S\Gamma B \cup B\Gamma S \cup (S\Gamma B)\Gamma S]] = (B \cup S\Gamma B \cup B\Gamma S \cup (S\Gamma B)\Gamma S]$. So $x \in (B)_T$.

Case 2: $z \in S\Gamma A$. Then $z = s\gamma c$ for some $s \in S$, $\gamma \in \Gamma$ and $c \in A$. If $c \neq a$, then $z = s\gamma c \in S\Gamma B \subseteq (B)_T$. Since $x \leq z$ and $z \in (B)_T$, we have $x \in (B)_T$. If $c = a$, then $z = s\gamma a \in S\Gamma(S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]$. Since $(S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]$ is a Γ-ideal of S for all $b \in S$, $z \in (S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S] \subseteq (B \cup S\Gamma B \cup B\Gamma S \cup (S\Gamma B)\Gamma S] = (B)_T$. Since $x \leq z$ and $z \in (B)_T$, we have $x \in (B)_T$.

Case 3: $z \in A\Gamma S$. Then $z = c\gamma s$ for some $c \in A$, $\gamma \in \Gamma$ and $s \in S$. If $c \neq a$, then $z = c\gamma s \in B\Gamma S \subseteq (B)_T$. Since $x \leq z$ and $z \in (B)_T$, we have $x \in (B)_T$. If $c = a$, then $z = a\gamma s \in (S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]\Gamma S$. Since $(S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]$ is a Γ-ideal of S for all $b \in S$, $z \in (S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S] \subseteq (B)_T$. Since $x \leq z$ and $z \in (B)_T$, $x \in (B)_T$.

Case 4: $z \in (S\Gamma A)\Gamma S$. Then $z = (s_1\gamma c)\beta s_2$ for some $s_1, s_2 \in S$, $\gamma, \beta \in \Gamma$ and $c \in A$. If $c \neq a$, then $z = (s_1\gamma c)\beta s_2 \in (S\Gamma B)\Gamma S \subseteq (B)_T$. Since $x \leq z$ and $z \in (B)_T$, we have $x \in (B)_T$. If $c = a$, then $z = (s_1\gamma a)\beta s_2 \in (S\Gamma(S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S])\Gamma S$. Since $(S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]$ is a Γ-ideal of S for all $b \in S$, $z \in (S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]\Gamma S \subseteq (S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S] \subseteq (B)_T$. Since $x \leq z$ and $z \in (B)_T$, $x \in (B)_T$. Thus, $(A)_T \subseteq (B)_T$. By $S = (A)_T \subseteq (B)_T \subseteq S$, hence $(B)_T = S$. This is a contradiction. Therefore, $a = b$. □

4. **Main Results.**

In this section, the algebraic structure of an ordered LA-Γ-semigroup with left identity containing two-sided bases will be presented.

Theorem 4.1. *A non-empty subset A of an ordered LA-Γ-semigroup S with left identity, is a two-sided base of S if and only if A satisfies the following two conditions:*

(1) *for any $x \in S$ there exists $a \in A$ such that $x \leq_I a$;*
(2) *for any $a, b \in A$, if $a \neq b$, then neither $a \leq_I b$ nor $b \leq_I a$.*

Proof: Assume that A is a two-sided base of S. Then $S = (A)_T$. Let $x \in S$. Since $x \in S = (A \cup S\Gamma A \cup A\Gamma S \cup (S\Gamma A)\Gamma S]$, we have $x \leq y$ for some $y \in A \cup S\Gamma A \cup A\Gamma S \cup (S\Gamma A)\Gamma S$. There are four cases to consider.

Case 1: $y \in A$. Since $x \leq y$, by Lemma 3.1, we have that $x \leq_I y$.

Case 2: $y \in S\Gamma A$. Then $y = s\gamma a$ for some $s \in S$, $\gamma \in \Gamma$ and $a \in A$. By $y = s\gamma a \in S\Gamma a \subseteq (a)_T$, $S\Gamma y \subseteq S\Gamma(S\Gamma a) = (S\Gamma S)\Gamma(S\Gamma a) = (a\Gamma S)\Gamma(S\Gamma S) = (a\Gamma S)\Gamma S = (S\Gamma S)\Gamma a = S\Gamma a \subseteq (a)_T$, $y\Gamma S \subseteq (S\Gamma a)\Gamma S \subseteq (a)_T$ and $(S\Gamma y)\Gamma S \subseteq (S\Gamma(S\Gamma a))\Gamma S = ((S\Gamma S)\Gamma(S\Gamma a))\Gamma S = ((a\Gamma S)\Gamma(S\Gamma S))\Gamma S = ((a\Gamma S)\Gamma S)\Gamma S = ((S\Gamma S)\Gamma a)\Gamma S = (S\Gamma a)\Gamma S \subseteq (a)_T$. Then $y \cup S\Gamma y \cup y\Gamma S \cup (S\Gamma y)\Gamma S \subseteq (a)_T$, and so $(y)_T = (y \cup S\Gamma y \cup y\Gamma S \cup (S\Gamma y)\Gamma S] \subseteq ((a)_T] = (a)_T$, i.e., $y \leq_I a$. Since $x \leq y$, by Lemma 3.1, we have $x \leq_I y$. So $x \leq_I y \leq_I a$. Thus, $x \leq_I a$.

Case 3: $y \in A\Gamma S$. Then $y = a\gamma s$ for some $a \in A$, $\gamma \in \Gamma$ and $s \in S$. By $y = a\gamma s \in a\Gamma S \subseteq (a)_T$, $S\Gamma y \subseteq S\Gamma(a\Gamma S) = a\Gamma(S\Gamma S) = a\Gamma S \subseteq (a)_T$, $y\Gamma S \subseteq (a\Gamma S)\Gamma S = (S\Gamma S)\Gamma a = S\Gamma a \subseteq (a)_T$ and $(S\Gamma y)\Gamma S \subseteq (S\Gamma(a\Gamma S))\Gamma S = (a\Gamma(S\Gamma S))\Gamma S = (a\Gamma S)\Gamma S = (S\Gamma S)\Gamma a = S\Gamma a \subseteq (a)_T$. Then $y \cup S\Gamma y \cup y\Gamma S \cup \Gamma S \subseteq (a)_T$, and so $(y)_T = (y \cup S\Gamma y \cup y\Gamma S \cup (S\Gamma y)\Gamma S] \subseteq ((a)_T] = (a)_T$, i.e., $y \leq_T a$. Since $x \leq y$, by Lemma 3.1, we have $x \leq_I y$. So $x \leq_I y \leq_I a$. Thus, $x \leq_I a$.

Case 4: $y \in (S\Gamma A)\Gamma S$. Then $y = (s_1\gamma a)\beta s_2$ for some $s_1, s_2 \in S$, $\gamma, \beta \in \Gamma$ and $a \in A$. By $y = (s_1\gamma a)\beta s_2 \in (S\Gamma a)\Gamma S \subseteq (a)_T$, $S\Gamma y \subseteq S\Gamma((S\Gamma a)\Gamma S) = (S\Gamma a)\Gamma(S\Gamma S) = (S\Gamma a)\Gamma S \subseteq (a)_T$, $y\Gamma S \subseteq ((S\Gamma a)\Gamma S)\Gamma S = (S\Gamma S)\Gamma(S\Gamma a) = (a\Gamma S)\Gamma(S\Gamma S) = (a\Gamma S)\Gamma S = (S\Gamma S)\Gamma a = S\Gamma a \subseteq (a)_T$ and $(S\Gamma y)\Gamma S \subseteq (S\Gamma((S\Gamma a)\Gamma S))\Gamma S = ((S\Gamma a)\Gamma(S\Gamma S))\Gamma S = ((S\Gamma a)\Gamma S)\Gamma S = (S\Gamma S)\Gamma(S\Gamma a) = (a\Gamma S)\Gamma(S\Gamma S) = (a\Gamma S)\Gamma S = (S\Gamma S)\Gamma a = S\Gamma a \subseteq (a)_T$. Then $y \cup S\Gamma y \cup y\Gamma S \cup (S\Gamma y)\Gamma S \subseteq (a)_T$, and so $(y)_T = (y \cup S\Gamma y \cup y\Gamma S \cup (S\Gamma y)\Gamma S] \subseteq ((a)_T] = (a)_T$, i.e., $y \leq_I a$. Since $x \leq y$, by Lemma 3.1, we have $x \leq_I y$. So $x \leq_I y \leq_I a$. Thus, $x \leq_I a$.

Hence the condition (1) holds. Next, let $a, b \in A$ such that $a \neq b$. Suppose $a \leq_I b$. Set $B = A\backslash\{a\}$. Then $b \in B$ and $B \subseteq A$. Let $x \in S$. By condition (1), there exists $c \in A$ such that $x \leq_I c$, i.e., $(x)_T \subseteq (c)_T$. There are two cases to consider. If $c \neq a$, then $c \in B$. So $x \in (x)_T \subseteq (c)_T \subseteq (B)_T$. If $c = a$, then $x \leq_I a \leq b_I$ and $x \leq_I b$, i.e., $(x)_T \subseteq (b)_T$. So $x \in (x)_T \subseteq (b)_T \subseteq (B)_T$. Thus, $S \subseteq (B)_T$ and so $S = (B)_T$. This is a contradiction. Hence $a \leq_I b$ is false. The case $b \leq_I a$ proved similarly. Hence the condition (2) holds.

Conversely, assume that the conditions (1) and (2) hold. We will show that A is a two-sided base of S. To show that $S = (A)_T$, let $x \in S$, by condition (1), there exists $a \in A$ such that $x \leq_I a$. Then $x \in (x)_T \subseteq (a)_T \subseteq (A)_T$. So $S \subseteq (A)_T$ and clearly $(A)_T \subseteq S$. Thus, $S = (A)_T$. Next, to show that A is a minimal subset of S with the property $S = (A)_T$, let $B \subset A$ such that $S = (B)_T$. Then there exists $a \in A$ and $a \notin B$. Since $a \in A$, $a \in S = (B)_T$. We will show that $a \notin (B]$. If $a \in (B]$, then $a \leq y$ for some $y \in B$, by Lemma 3.1, $a \leq_I y$. This is a contradiction. So $a \notin (B]$. Thus, $a \in (S\Gamma B \cup B\Gamma S \cup (S\Gamma B)\Gamma S]$. Since $a \in (S\Gamma B \cup B\Gamma S \cup (S\Gamma B)S]$, we have $a \leq c$ for some $c \in S\Gamma B \cup B\Gamma S \cup (S\Gamma B)\Gamma S$. There are three cases to consider.

Case 1: $c \in S\Gamma B$. Then $c = s\gamma b_1$ for some $s \in S$, $\gamma \in \Gamma$ and $b_1 \in B$. Since $a \leq c$ and $c = s\gamma b_1 \in S\Gamma b_1 \subseteq b_1 \cup S\Gamma b_1 \cup b_1\Gamma S \cup (S\Gamma b_1)\Gamma S$, $a \in (b_1 \cup S\Gamma b_1 \cup b_1\Gamma S \cup (S\Gamma b_1)\Gamma S] = (b_1)_T$. It follows that $(a)_T \subseteq (b_1)_T$. Thus, $a \leq_I b_1$ where $a, b_1 \in A$. This is a contradiction.

Case 2: $c \in B\Gamma S$. Then $c = b_2\gamma s$ for some $s \in S$, $\gamma \in \Gamma$ and $b_2 \in B$. Since $a \leq c$ and $c = b_2\gamma s \in b_2\Gamma S \subseteq b_2 \cup S\Gamma b_2 \cup b_2\Gamma S \cup (S\Gamma b_2)\Gamma S$, $a \in (b_2 \cup S\Gamma b_2 \cup b_2\Gamma S \cup (S\Gamma b_2)\Gamma S] = (b_2)_T$. It follows that $(a)_T \subseteq (b_2)_T$. Thus, $a \leq_I b_2$ where $a, b_2 \in A$. This is a contradiction.

Case 3: $c \in (S\Gamma B)\Gamma S$. Then $c = (s_1\gamma b_3)\beta s_2$ for some $s_1, s_2 \in S$, $\gamma, \beta \in \Gamma$ and $b_3 \in B$. Since $a \leq c$ and $c = (s_1\gamma b_3)\beta s_1 \in (S\Gamma b_3)\Gamma S \subseteq b_3 \cup S\Gamma b_3 \cup b_3\Gamma S \cup (S\Gamma b_3)\Gamma S$, $a \in (b_3 \cup$

$S\Gamma b_3 \cup b_3\Gamma S \cup (S\Gamma b_3)\Gamma S] = (b_3)_T$. It follows that $(a)_T \subseteq (b_3)_T$. Thus, $a \leq_I b_3$ where $a, b_3 \in A$. This is a contradiction.

Therefore, A is a two-sided base of S. The proof is completed. □

Theorem 4.2. *Let A be a two-sided base of an ordered LA-Γ-semigroup S with left identity, such that $(a)_T = (b)_T$, for some a in A and b in S. If $a \neq b$, then S contains at the least two two-sided bases.*

Proof: Assume that $a \neq b$. Suppose that $b \in A$. Since $a \neq b$ and $a \in (a)_T = (b)_T = (b \cup S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S] = (b] \cup (S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]$, $a \in (b]$ or $a \in (S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]$. If $a \in (b]$, then $a \leq b$, by Lemma 3.1, we have $a \leq_I b$ where $a, b \in A$. This is a contradiction. So $a \in (S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S]$. By Lemma 3.2, $a = b$. This is a contradiction. Thus, $b \in S\backslash A$. Setting $B = (A\backslash\{a\}) \cup \{b\}$, then $B \neq A$. We will show that B is a two-sided base of S using Theorem 4.1. First, let $x \in S$. Since A is a two-sided base of S, by Theorem 4.1(1), $x \leq_I c$ for some $c \in A$. If $c \neq a$, then $c \in B$. If $c = a$, then $(c)_T = (a)_T$. Since $(a)_T = (b)_T$, we have $(c)_T = (b)_T$, i.e., $c \leq_I b$. So $x \leq_I c \leq_I b$. Thus, $x \leq_I b$ where $b \in B$. Next, let $c_1, c_2 \in B$ such that $c_1 \neq c_2$. We will show that neither $c_1 \leq_I c_2$ nor $c_2 \leq_I c_1$. Then there are four cases to consider.

Case 1: $c_1 \neq b$ and $c_2 \neq b$. Then $c_1, c_2 \in A$. Since A is a two-sided base of S, then neither $c_1 \leq_I c_2$ nor $c_2 \leq_I c_1$.

Case 2: $c_1 \neq b$ and $c_2 = b$. Then $(c_2)_T = (b)_T$. If $c_1 \leq_I c_2$, then $(c_1)_T \subseteq (c_2)_T = (b)_T = (a)_T$. Thus, $c_1 \leq_I a$ where $c_1, a \in A$. This is contradiction. If $c_2 \leq_I c_1$, then $(a)_T = (b)_T = (c_2)_T \subseteq (c_1)_T$. Thus, $a \leq_I c_1$ where $c_1, a \in A$. This is a contradiction.

Case 3: $c_1 = b$ and $c_2 \neq b$. Then $(c_1)_T = (b)_T$. If $c_1 \leq_I c_2$, then $(a)_T = (b)_T = (c_1)_T \subseteq (c_2)_T$. Thus, $a \leq_I c_2$ where $c_2, a \in A$. This is contradiction. If $c_2 \leq_I c_1$, then $(c_2)_T \subseteq (c_1)_T = (b)_T = (a)_T$. Thus, $c_2 \leq_I a$ where $c_2, a \in A$. This is a contradiction.

Case 4: $c_1 = b$ and $c_2 = b$. This is impossible.

Therefore, B is a two-sided base of S. □

The following corollary follows directly from Theorem 4.2.

Corollary 4.1. *Let A be a two-sided base of an ordered LA-Γ-semigroup S with left identity, and let $a \in A$. If $(x)_T = (a)_T$ for some $x \in S$, $x \neq a$, then x belongs to some two-sided base of S, which is different from A.*

Theorem 4.3. *Let A and B be two-sided bases of ordered LA-Γ-semigroup S with left identity. Then A and B have the same cardinality.*

Proof: Let A and B be two-sided bases of S. Let $a \in A$. Since B is a two-sided base of S, by Theorem 4.1(1), there exists $b \in B$ such that $a \leq_I b$. Similarly, since A is a two-sided base of S, there exists $a^* \in A$ such that $b \leq_I a^*$. So $a \leq_I b \leq_I a^*$, and $a \leq_I a^*$. By Theorem 4.1(2), $a = a^*$. Hence $(a)_T = (b)_T$. Now, define a mapping $\varphi : A \to B$; $\varphi(a) = b$ for all $a \in A$. First, to show that φ is well-defined, let $a_1, a_2 \in A$ such that $a_1 = a_2$, $\varphi(a_1) = b_1$, and $\varphi(a_2) = b_2$ for some $b_1, b_2 \in B$. Then $(a_1)_T = (b_1)_T$ and $(a_2)_T = (b_2)_T$. Since $a_1 = a_2$, $(a_1)_T = (a_2)_T$. Thus, $(a_1)_T = (a_2)_T = (b_1)_T = (b_2)_T$, so $b_1 \leq_I b_2$ and $b_2 \leq_I b_1$. By Theorem 4.1(2), $b_1 = b_2$. Hence $\varphi(a_1) = \varphi(a_2)$. Therefore, φ is well-defined. Next, to show that φ is one-to-one, let $a_1, a_2 \in A$ such that $\varphi(a_1) = \varphi(a_2)$. Then $\varphi(a_1) = \varphi(a_2) = b$ for some $b \in B$. We have $(a_1)_T = (a_2)_T = (b)_T$. Since $(a_1)_T = (a_2)_T$, $a_1 \leq_I a_2$ and $a_2 \leq_I a_1$. Thus, $a_1 = a_2$. Therefore, φ is one-to-one. Finally, to show that φ is onto, let $b \in B$, and then there exists $a \in A$ such that $b \leq_I a$. Similarly, there exists $b^* \in B$ such that $a \leq_I b^*$. Then $b \leq_I a \leq_I b^*$, i.e., $b \leq_I b^*$. By Theorem 4.1(2), $b = b^*$. So $b \leq_I a$ and $a \leq_I b$, i.e., $(b)_T = (a)_T$ and $(a)_T = (b)_T$. Thus, $(a)_T = (b)_T$. Therefore, φ is onto. This completes the proof. □

If a two-sided base of an ordered LA-Γ-semigroup S with left identity, is a Γ-ideal of S, then $S = (A \cup S\Gamma A \cup A\Gamma S \cup (S\Gamma A)\Gamma S] \subseteq (A \cup A \cup A \cup A] = (A] = A$. Hence $S = A$. The converse statement is obvious. Then we conclude that a two-sided base A of an ordered LA-Γ-semigroup S with left identity, is a Γ-ideal of S if and only if $A = S$.

In Example 3.1, it is observed that not every two-sided base of an ordered LA-Γ-semigroup S with left identity, is an LA-Γ-subsemigroup. The following theorem gives necessary and sufficient conditions of a two-sided base of an ordered LA-Γ-semigroup S with left identity, to be an LA-Γ-subsemigroup.

Theorem 4.4. *A two-sided base A of an ordered LA-Γ-semigroup S with left identity, is an LA-Γ-subsemigroup if and only if $A = \{a\}$ with $a\gamma a = a$ for all $\gamma \in \Gamma$.*

Proof: Assume that A is an LA-Γ-subsemigroup of S. Let $a, b \in A$ and $\gamma \in \Gamma$. Since A is an LA-Γ-subsemigroup S, we have $a\gamma b \in A$. Set $a\gamma b = c$. Then $c = a\gamma b \in S\Gamma b \subseteq (S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S]$. By Lemma 3.2, we have $c = b$. So $a\gamma b = b$. Similarly, $c = a\gamma b \in a\Gamma S \subseteq (S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]$. By Lemma 3.2, we have $c = a$. So $a\gamma b = a$. Thus, $a = b$. Therefore, $A = \{a\}$ with $a\gamma a = a$. The converse statement is clear. □

The union of all two-sided bases of an ordered LA-Γ-semigroup S with left identity is denoted by C.

Theorem 4.5. *Let S be an ordered LA-Γ-semigroup with left identity. Then $S\backslash C = \varnothing$ or a Γ-ideal of S.*

Proof: Assume that $S\backslash C \neq \varnothing$. We will show that $S\backslash C$ is a Γ-ideal of S. Let $x \in S$, $\gamma \in \Gamma$ and $a \in S\backslash C$. To show that $x\gamma a \in S\backslash C$ and $a\gamma x \in S\backslash C$, suppose that $x\gamma a \notin S\backslash C$. Then $x\gamma a \in C$. Thus, $x\gamma a \in A$ for a two-sided base A of S. Let $x\gamma a = b$ for some $b \in A$. Since $b = x\gamma a \in S\Gamma a \subseteq (a)_T$, $b \in (a)_T$. It follows that $(b)_T \subseteq (a)_T$. If $(b)_T = (a)_T$, by Corollary 4.1, we have that $a \in C$. This is a contradiction. Thus, $(b)_T \subset (a)_T$, i.e., $b <_I a$. Since A is a two-sided base of S, by Theorem 4.1(1), there exists $b_1 \in A$ such that $a \leq b_1$. Since $b <_I a \leq_I b_1$, $b \leq_I b_1$ where $b, b_1 \in A$. This is a contradiction. Thus, $x\gamma a \in S\backslash C$. Similarly, we can show that $a\gamma x \in S\backslash C$. Next, to show that if $a_1 \in S\backslash C$ and $a_2 \in S$ such that $a_2 \leq a_1$, then $a_2 \in S\backslash C$. Suppose that $a_2 \in C$. Then $a_2 \in B$ for a two-sided base B of S. Since B is a two-sided base of S, by Theorem 4.1(1), there exists $a_3 \in B$ such that $a_1 \leq_I a_3$. Since $a_2 \leq a_1$, by Lemma 3.1, $a_2 \leq_I a_1$. We have that $a_2 \leq_I a_3$ where $a_2, a_3 \in B$. This is a contradiction. Thus, $a_2 \notin C$, i.e., $a_2 \in S\backslash C$. Therefore, $S\backslash C$ is a Γ-ideal of S. □

Let M^* be a proper Γ-ideal of an ordered LA-Γ-semigroup S with left identity, containing every proper Γ-ideal of S.

Theorem 4.6. *Let S be an ordered LA-Γ-semigroup with left identity, and $\varnothing \neq C \subset S$. Then $S\backslash C = M^*$ if and only if every two-sided base of S is one-element base.*

Proof: Assume that $S\backslash C = M^*$. Then $S\backslash C$ is a maximal proper Γ-ideal of S. We will show that for every $a \in C$, $C \subseteq (a)_T$. Let $a \in C$. Suppose $C \nsubseteq (a)_T$. Since $C \nsubseteq (a)_T$ and $\varnothing \neq C \subset S$, $(a)_T$ is a proper Γ-ideal of S. Thus, $a \in (a)_T \subseteq M^* = S\backslash C$, and so $a \in S\backslash C$, i.e., $a \notin C$. This is a contradiction. Hence $C \subseteq (a)_T$ for every $a \in C$. We will show that for every $a \in C$, $S\backslash C \subseteq (a)_T$. Suppose that $S\backslash C \not\subset (a^*)_T$ for some $a^* \in C$. Then $(a^*)_T \neq S$, and so $(a^*)_T$ is a proper Γ-ideal of S. Thus, $a^* \in (a^*)_T \subseteq M^* = S\backslash C$, and so $a^* \in S\backslash C$, i.e., $a^* \notin C$. This is a contradiction. Hence $S\backslash C \subseteq (a)_T$ for every $a \in C$. Since $S\backslash C \subseteq (a)_T$ and $C \subseteq (a)_T$ for every $a \in C$, we have $S = (S\backslash C) \cup C \subseteq (a)_T \subseteq S$. So $S = (a)_T$ for every $a \in C$. Thus, $\{a\}$ is a two-sided base of S. Next, let A be a two-sided base of S. We will show that $a = b$ for every $a, b \in A$. Suppose that there exists $a, b \in A$ such that $a \neq b$. Since A is a two-sided base of S, $a \in A \subseteq C$ and $a \in C$. So $S = (a)_T$. Since $a \neq b$ and $b \in S = (a \cup S\Gamma a \cup a\Gamma S \cup (S\Gamma a)\Gamma S] = (a] \cup (S\Gamma a \cup a\Gamma S \cup (S\Gamma a)\Gamma S]$, $b \in (a]$ or $b \in (S\Gamma a \cup a\Gamma S \cup (S\Gamma a)\Gamma S]$. If $b \in (a]$, then $b \leq a$ by Lemma 3.1, $b \leq_I a$. This is a contradiction. So $b \in (S\Gamma a \cup a\Gamma S \cup (S\Gamma a)\Gamma S]$. By Lemma 3.2, $a = b$. This is a contradiction. Therefore, every two-sided base of S is one-element base.

Conversely, assume that every two-sided base of S is a one-element base. Then $S = (a)_T$ for every $a \in C$. To show that $S\backslash C = M^*$, since $\varnothing \neq C \subset S$, $\varnothing \neq S\backslash C \subset S$. By Theorem 4.5, $S\backslash C$ is a proper Γ-ideal of S. Next, let M be a proper Γ-ideal of S such that $S\backslash C \subset M \subset S$. Since $S\backslash C \subset M$, there exists $x \in M$ such that $x \notin S\backslash C$, i.e., $x \in C$. We have $x \in M \cap C$. So $M \cap C \neq \varnothing$. Let $b \in M \cap C$. Since $b \in M$, $S\Gamma b \subseteq S\Gamma M \subseteq M$, $b\Gamma S \subseteq M\Gamma S \subseteq M$ and $(S\Gamma b)\Gamma S \subseteq (S\Gamma M)\Gamma S \subseteq M\Gamma S \subseteq M$, $b \cup S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S \subseteq M$. We have $(b)_T = (b \bigcup S\Gamma b \cup b\Gamma S \bigcup (S\Gamma b)\Gamma S] \subseteq (M] = M$. Since $b \in C$, by assumption, we have $(b)_T = S$. So $S = (b)_T \subseteq M \subset S$. Thus, $M = S$. This is a contradiction. Hence $S\backslash C$ is a maximal proper Γ-ideal of S. Finally, let B be a Γ-ideal of S such that $B \nsubseteq S\backslash C$. Since $B \nsubseteq S\backslash C$, there exists $x \in B$ such that $x \notin S\backslash C$, i.e., $x \in C$. So $B \cap C \neq \varnothing$. Let $c \in B \cap C$. Since $c \in B$, $S\Gamma c \subseteq S\Gamma B \subseteq B$, $c\Gamma S \subseteq B\Gamma S \subseteq B$ and $(S\Gamma c)\Gamma S \subseteq (S\Gamma B)\Gamma S \subseteq B\Gamma S \subseteq B$, $c \cup S\Gamma c \cup c\Gamma S \cup (S\Gamma c)\Gamma S \subseteq B$. We have $(c)_T = (c \cup S\Gamma c \cup c\Gamma S \cup (S\Gamma c)\Gamma S] \subseteq (B] = B$. Since $c \in C$, $S = (c)_T \subseteq B \subseteq S$. Thus, $S = B$. Therefore, $S\backslash C = M^*$. □

Theorem 4.7. *Let S be an ordered LA-Γ-semigroup with left identity. If e is a left identity of S, then $\{e\}$ is a two-sided base of S.*

Proof: Assume that e is a left identity of S. Let $A = \{e\}$. We will show that A is a two-sided base of S. To show that $S = (A)_T$, since e is a left identity of S, by Lemma 2.1, we have $S = e\Gamma S = S\Gamma e$. Since $S = S\Gamma e$, we have $(S\Gamma e)\Gamma S = (S\Gamma e)\Gamma(S\Gamma e) = (S\Gamma S)\Gamma(e\Gamma e) = S\Gamma e$. So $e \cup S\Gamma e \cup e\Gamma S \cup (S\Gamma e)\Gamma e = S$. Thus, $(A)_T = (e \cup S\Gamma e \cup e\Gamma S \cup (S\Gamma e)\Gamma S] = (S] = S$. Hence $(A)_T = S$. Clearly, A is a minimal subset of S with the property $S = (A)_T$. Therefore, A is a two-sided base of S. □

In Examples 3.1 and 3.2, it is observed that every two-sided base of an ordered LA-Γ-semigroup with left identity is one-element base. This leads to proving the following corollary. From Theorem 4.3 and Theorem 4.7, we can easily obtain the following result.

Corollary 4.2. *Let S be an ordered LA-Γ-semigroup with left identity. Then every two-sided base of S is one-element base.*

In Example 3.2, we have the all two-sided bases of S are $A_1 = \{c\}$, $A_2 = \{d\}$ and $A_3 = \{e\}$. Then $S\backslash C = \{a, b\}$ is a maximal proper Γ-ideal of S containing every proper Γ-ideal of S. We have the following result is combining Theorem 4.6 and Corollary 4.2.

Theorem 4.8. *Let S be an ordered LA-Γ-semigroup with left identity. Then $S\backslash C$ is a maximal proper Γ-ideal of S containing all proper Γ-ideals of S.*

Proof: Let S be an ordered LA-Γ-semigroup with left identity. By Corollary 4.2, we have every two-sided base of S is one-element base. Since every two-sided base of S is one element base, by Theorem 4.6, we obtain $S\backslash C = M^*$. Therefore, $S\backslash C$ is a maximal proper Γ-ideal of S containing all proper Γ-ideals of S. □

5. **Conclusion.** In this paper, we focus on the results for two-sided bases of ordered LA-Γ-semigroups with left identity. We show in Corollary 4.2 that every two-sided base of an ordered LA-Γ-semigroup with left identity is one-element base. Finally, we prove in Theorem 4.8 that the complement of union of all two-sided base of an ordered LA-Γ-semigroup with left identity is the maximal proper Γ-ideal. In the future work, we can study other results in this algebraic structures. Moreover, we may use the essential (m, n)-ideal of semigroups defined in [10] to define essential (m, n)-bases of semigroups and study their properties.

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