(m,n) ควอร์ซีไอดีลบนกึ่งกรุปเกือบทางซ้ายอันดับ

ON (m,n) QUASI-IDEALS IN ORDERED LA-SEMIGROUPS

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บทคัดย่อ

ในบทความวิจัยนี้ศึกษาแนวคิดของ (m,n) ควอร์ซีไอดีลบนกึ่งกรุปเกือบทางซ้ายอันดับ โดยศึกษาคล้ายกับแนวคิดของ (m,n) ควอร์ซีไอดีลบนกึ่งกรุปเกือบทางซ้าย ซึ่ง ธิติ เกตุคำ ได้ทำการศึกษาในปี ค.ศ. 2015 นอกจากนี้ยังแนะนำแนวคิดของ (m,n) ควอร์ซีไอดีบบนกึ่งกรุปเกือบทางซ้ายอันดับ และอธิบายคุณสมบัติบางประการของ (m,n) ควอร์ซีไอดีบบนกึ่งกรุปเกือบทางซ้ายอันดับ พร้อมทั้งศึกษาความสัมพันธ์ของ (m,n) ควอร์ซีไอดีบบนกึ่งกรุปเกือบทางซ้ายอันดับปรกติ
คำสำคัญ: กึ่งกรุปเกือบทางซ้ายอันดับ, (m,n)-ควอร์ซีไอดีล, m-ไอดีลซ้าย, n-ไอดีลขวา

ABSTRACT

The aim of this paper is to study the concept of (m,n)-quasi-ideals in ordered LA-semigroups that are studied analogously to the concept of (m,n)-quasi-ideals in LA-semigroups considered by Thiti Gaketem in 2015. We introduce the notion of (m,n)-quasi-ideals in ordered LA-semigroups, and describe some property of (m,n)-quasi-ideals in ordered LA-semigroup. Also including the study relations of (m,n)-quasi-ideals in regular ordered LA-semigroups.
Keywords: ordered LA-semigroups, (m,n)-quasi-ideals, m-left ideals, n-right ideals

1. Introduction

The notion of quasi-ideal of a semigroup was introduced by R. Chinram and R. Sripakorn [4]. Ansari, Khan and Kaushik [2] characterized the notion of (m,n)-quasi-ideals in semigroups.

The concept of an AG-groupoid was first given by Kazim and Naseeruddin [5] in 1972 and they called it left almost semigroups (LA-semigroups). An LA-semigroup is a useful algebraic structure, midway between a groupoid and a commutative semigroup. An LA-semigroup is non-associative in general, however, there is a close relationship with semigroup as well as with commutative structures.

The concept of an ordered LA-semigroup was first given by Shah et al. [9] and then Khan and Faisal in [6],[11],[12], applied theory of fuzzy sets to ordered LA-semigroup.

In this paper we study (m,n)-quasi-ideals in ordered LA-semigroups. We generalize some facts of m-left ideals, n-right ideals of ordered LA-semigroups and study the properties of (m,n)-quasi-ideals in ordered LA-semigroups.

2. Preliminaries and basic definitions

Before going to prove the main results we need the following definitions that we use later.

Definition 2.1 [1] A groupoid $(S,\cdot)$ is called an LA-semigroup or an AG-groupoid, if it satisfies left invertive law $(a\cdot b)\cdot c=(c\cdot b)\cdot a,$ for all $a,b,c\in S.$

Lemma 2.1 [7] In an LA-semigroup S it satisfies the medial law if

$$(ab)(cd)=(ac)(bd), \text{ for all } a,b,c,d\in S.$$

Definition 2.2 [8] An element $e\in S$ is called left identity if $ea=a$ for all $a\in S.$

Lemma 2.2 [1] If S is an LA-semigroup with left identity, then $a(bc)=b(ac),$ for all $a,b,c\in S.$

Lemma 2.3 [1] If S is an LA-semigroup with left identity $e,$ then $SS=S$ and $S=eS=Se.$

Lemma 2.4 [7] An LA-semigroup S with left identity it satisfies the paramedial if

$$(ab)(cd)=(dc)(ba), \text{ for all } a,b,c,d\in S.$$

Definition 2.3 [1] An LA-semigroup S is called a locally associative LA-semigroup if it satisfies

$$(aa)a=a(aa), \text{ for all } a\in S.$$

Theorem 2.1 [1] Let S be a locally associative LA-semigroup then $a^1=a$ and $a^{n+1}=a^n a,$ for $n\geq 1;$ for all $a\in S.$

Theorem 2.2 [1] Let S be a locally associative LA-semigroup with left identity then $a^m a^n=a^{m+n},(a^m)^n=a^{mn}$ and $(ab)^n=a^n b^n,$ for all $a,b\in S$ and m,n are positive integer.

Definition 2.4 [6] An ordered LA-semigroup (po-LA-semigroup) is a structure $(S,\cdot,\leq)$ in which the following conditions hold:

(i) $(S,\cdot)$ is an LA-semigroup,

(ii) $(S,\leq)$ is a poset (reflexive, anti-symmetric and transitive),

(iii) for all a,b and $x\in S,$ $a\leq b$ implies $ax\leq bx$ and $xa\leq xb.$

Throughout this article, unless stated otherwise, S stands for an ordered LA-semigroup.

For a non-empty subset A and B of an ordered LA-semigroup $S,$ we define

$$AB=\{ab \mid a\in A \text{ and } b\in B\}$$

and

$$(A]=\{x\in S \mid x\leq a \text{ for some } a\in A\}.$$

For $A=\{a\},$ we usually write it as $(a].$

Definition 2.5 [6] A non-empty subset A of an ordered LA-semigroup $S,$ is called an LA-subsemigroup of $S,$ if $A^2\subseteq A.$

Definition 2.6 [6] A non-empty subset A of an ordered LA-semigroup S is called a left (right) ideal of $S,$ if

(i) $SA\subseteq A$ $(AS\subseteq A),$

(ii) If $a\in A$ and $b\in S$ such that $b\leq a,$ then $b\in A.$

A non-empty subset A of an ordered LA-semigroup S is called a two sided ideal of S if it is both a left and a right ideal of S.

Definition 2.7 [6] A non-empty subset A of an ordered LA-semigroup S is called a quasi-ideal of $S,$ if

(i) $(AS]\cap(SA]\subseteq A,$

(ii) If $a\in A$ and $b\in S$ such that $b\leq a,$ then $b\in A.$

Lemma 2.5 [11] In an ordered LA-semigroup $S,$ the following are true:

(i) $A\subseteq(A],\ \forall A\subseteq S.$ (ii) $A\subseteq B\Rightarrow(A]\subseteq(B],\ \forall A,B\subseteq S.$ (iii) $(A](B]\subseteq(AB],\ \forall A,B\subseteq S.$

(iv) $((A]]=(A],\ \forall A\subseteq S.$ (v) $((A](B]]=(AB],\ \forall A,B\subseteq S.$

(vi) $(A\cap B]\subseteq(A]\cap(B],\ \forall A,B\subseteq S.$ (vii) $(A\cup B]=(A]\cup(B],\ \forall A,B\subseteq S.$

3. (m,n)-quasi-ideals in ordered LA-semigroups

In this section we define and study (m,n)-quasi-ideals of an ordered LA-semigroup in a similar manner to (m,n)-quasi-ideals of LA-semigroups.

Definition 3.1 A nonempty subset A of an ordered LA-semigroup S is called m-left (n-right) ideal of S if

(i) $S^mA\subseteq A$ (resp. $AS^n\subseteq A$),

(ii) If $a\in A$ and $b\in S$ such that $b\leq a,$ then $b\in A.$

Definition 3.2 A nonempty subset A of an ordered LA-semigroup S is called an (m,n)-quasi-ideal of S if

(i) $(S^mA]\cap(AS^n]\subseteq A,$ where m,n are positive integers,

(ii) If $a\in A$ and $b\in S$ such that $b\leq a,$ then $b\in A.$

Example 3.1 [3] Let $S=\{a,b,c,d\}$ be an ordered LA-semigroup with the multiplication table and order below:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	a	b
d	a	a	b	c

$$\leq=\{(a,a),(a,b),(a,c),(b,b),(c,c)\}$$

Let $Q=\{a\}.$ we have that $S^2Q\cap QS^1=\{a\}\cap\{a\}=\{a\}=Q.$ This implies that Q is a $(2,1)$-quasi-ideal of $S.$ Let $A=\{a,c\}.$ We have that $S^2A=\{a\}\subseteq\{a,c\}=A.$ Hence A is a 2-left-ideal of $S.$ Let $B=\{a,b,c\}.$ We have that $BS^1=\{a,b\}\subseteq\{a,b,c\}=B.$ This implies that B is a 1-right-ideal of $S.$

Lemma 3.1 Let S be an ordered LA-semigroup and let T_i be an LA-subsemigroup of S for all $i\in I.$ If $\bigcap_{i\in I}T_i\neq\varnothing,$ then $\bigcap_{i\in I}T_i$ is an LA-subsemigroup.

Proof. Assume that $\bigcap_{i\in I} T_i \neq \varnothing$. Let $a,b \in \bigcap_{i\in I} T_i$ for all $i \in I$. Since T_i is an LA-subsemigroup for all $i \in I$, we have $ab \in T_i$ for all $i \in I$. Hence $ab \in \bigcap_{i\in I} T_i$. Thus $\bigcap_{i\in I} T_i$ is an LA-subsemigroup.

Theorem 3.2 Let S be an ordered LA-semigroup and let Q_i be an (m,n)-quasi-ideal of S for all $i \in I$. If $\bigcap_{i\in I} Q_i \neq \varnothing$, then $\bigcap_{i\in I} Q_i$ is an (m,n)-quasi-ideal.

Proof. Assume that $\bigcap_{i\in I} Q_i \neq \varnothing$. By Lemma 3.1, we have that $\bigcap_{i\in I} Q_i$ is an LA-subsemigroup of S. Consider $(S^m(\bigcap_{i\in I} Q_i)] \cap ((\bigcap_{i\in I} Q_i)S^n] \subseteq (S^m Q_i] \cap (Q_i S^n] \subseteq Q_i$ for all $i \in I$. Hence $(S^m(\bigcap_{i\in I} Q_i)] \cap ((\bigcap_{i\in I} Q_i)S^n] \subseteq \bigcap_{i\in I} Q_i$. If $x \in \bigcap_{i\in I} Q_i$ and $y \in S$ such that $y \leq x$, then $y \in Q_i$ for all $i \in I$. Therefore $y \in \bigcap_{i\in I} Q_i$. Hence $\bigcap_{i\in I} Q_i$ is an (m,n)-quasi-ideal of S.

Definition 3.3 A subset A of an ordered LA-semigroup S is called an $(m,0)$-ideal ($(0,n)$-ideal) of S if

(i) $SA^m \subseteq A$ $(A^n S \subseteq A)$ for $m,n \in \mathbb{N}$,

(ii) If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Lemma 3.3 Let S be an ordered LA-semigroup with left identity and $a \in S$. Then the following statements hold true:

(i) $(S^m a]$ is an m-left ideal of S.

(ii) $(a^2 S^n]$ is an n-right ideal of S.

(iii) $(S^m a] \cap (a^2 S^n]$ is an (m,n)-quasi-ideal.

Proof. (i) We get $(S^m a](S^m a] \subseteq ((S^m a)(S^m a)] \subseteq ((S^m S)(S^m a)] = ((aS^m)(SS^m)] = ((aS^m)S^m]$ $= ((S^m S^m)a] = (S^m a]$. Hence $(S^m a]$ is an LA-subsemigroup. First we will show that $(S^m a]$ is an m-left ideal of S, i.e. $S^m(S^m a] \subseteq (S^m a]$. Let $x \in S^m(S^m a]$ then $x = yb$ for some $y \in S^m$ and $b \in (S^m a]$, where $b \leq sa$ for some $s \in S^m$. Since $SS = S$, so let $y = z_1 z_2$. Thus $x \leq y(sa) = (z_1 z_2)(sa) = (as)(z_2 z_1) = ((z_2 z_1)s)a \subseteq S^m a$. Therefore $x \in (S^m a]$. For the second condition, let x be any element in $(S^m a]$, then $x \leq ba$ for some ba in $S^m a$. Let y be any other element of S such that $y \leq x \leq ba$, which implies that y is in $(S^m a]$. Hence $(S^m a]$ is an m-left ideal of S.

(ii) We get $(a^2 S^n](a^2 S^n] \subseteq ((a^2 S^n)(a^2 S^n)] \subseteq ((SS^n)(a^2 S^n)] = (S^n(a^2 S^n)] = (a^2(S^n S^n)] = (a^2 S^n]$ and $(a^2 S^n]S^n \subseteq (a^2 S^n](S^n] \subseteq ((a^2 S^n)S^n] = ((S^n S^n)a^2] = (a((S^n S^n)a)] = (a((aS^n)S^n)]$ $= ((aS^n)(aS^n)] = ((aa)(S^n S^n)] = (a^2 S^n]$. It is easy to see that $x \in (a^2 S^n]$ and $y \in S$ such that $y \leq x$, then $y \in (a^2 S^n]$. Hence $(a^2 S^n]$ is an n-right ideal of S.

(iii) We have $((S^m a] \cap (a^2 S^n])((S^m a] \cap (a^2 S^n]) \subseteq (S^m a]((S^m a] \cap (a^2 S^n])$ $= (S^m a](S^m a] \cap (S^m a](a^2 S^n] \subseteq ((S^m a)(S^m a)] \cap ((S^m a)(a^2 S^n)] \subseteq (S^m a] \cap ((S^m a)(a^2 S^n)] \subseteq (S^m a]$ and $((S^m a] \cap (a^2 S^n])((S^m a] \cap (a^2 S^n]) \subseteq (a^2 S^n]((S^m a] \cap (a^2 S^n]) = (a^2 S^n](S^m a] \cap (a^2 S^n](a^2 S^n]$ $\subseteq ((a^2 S^n)(S^m a)] \cap ((a^2 S^n)(a^2 S^n)] \subseteq ((a^2 S^n)(S^m a)] \cap (a^2 S^n)] \subseteq (a^2 S^n]$.

Combining these two $((S^m a] \cap (a^2 S^n])((S^m a] \cap (a^2 S^n]) \subseteq (S^m a] \cap (a^2 S^n]$. We obtain $(S^m((S^m a] \cap (a^2 S^n])] \cap (((S^m a] \cap (a^2 S^n])S^n] \subseteq ((S^m]((S^m a] \cap (a^2 S^n])] \cap (((S^m a] \cap (a^2 S^n])(S^n]]$

$= ((S^m](S^m a] \cap (S^m](a^2S^n]] \cap ((S^m a](S^n] \cap (a^2S^n](S^n]] \subseteq ((S^m(S^m a)] \cap (S^m(a^2S^n)]]$
$\cap(((S^m a)S^n] \cap ((a^2S^n)S^n]] \subseteq ((S^m a] \cap (S^m(a^2S^n)]] \cap (((S^m a)S^n] \cap (a^2S^n]] \subseteq ((S^m a]] \cap ((a^2S^n]]$
$= (S^m a] \cap (a^2S^n]$. Let $x \in (S^m a] \cap (a^2S^n]$ and $y \in S$ such that $y \le x$, we have $y \in (S^m a] \cap (a^2S^n]$. This shows that $(S^m a] \cap (a^2S^n]$ is an (m,n)-quasi-ideal.

Theorem 3.4 Let S be an ordered LA-semigroup. The following statements are true:
(i) Let L_i be an m-left ideal of S for all $i \in I$. If $\bigcap_{i\in I} L_i \ne \varnothing$, then $\bigcap_{i\in I} L_i$ is m-left ideal of S.
(ii) Let R_i be an n-right ideal of S for all $i \in I$. If $\bigcap_{i\in I} R_i \ne \varnothing$, then $\bigcap_{i\in I} R_i$ is n-right ideal of S.

Proof. (i) Since L_i be an m-left ideal of S for all $i \in I$, we have $S^m L_i \subseteq L_i$. We will show that $\bigcap_{i\in I} L_i$ is m-left ideal of S. Assume that $\bigcap_{i\in I} L_i \ne \varnothing$. By Lemma 3.1, we have $\bigcap_{i\in I} L_i$ is an LA-subsemigroup of S. Since $S^m(\bigcap_{i\in I} L_i) \subseteq S^m L_i \subseteq L_i$, we have $S^m(\bigcap_{i\in I} L_i) \subseteq \bigcap_{i\in I} L_i$. Let $x \in \bigcap_{i\in I} L_i$ and $y \in S$ such that $y \le x$, then $y \in \bigcap_{i\in I} L_i$. Hence $\bigcap_{i\in I} L_i$ is an m-left ideal of S.
(ii) Since R_i be an n-right ideal of S for all $i \in I$, we have $R_i S^n \subseteq R_i$. We will show that $\bigcap_{i\in I} R_i$ is n-right ideal of S. Assume that $\bigcap_{i\in I} R_i \ne \varnothing$. By Lemma 3.1, we have $\bigcap_{i\in I} R_i$ is an LA-subsemigroup of S. Since $(\bigcap_{i\in I} R_i)S^n \subseteq R_i S^n \subseteq R_i$, we have $(\bigcap_{i\in I} R_i)S^n \subseteq \bigcap_{i\in I} R_i$. Let $x \in \bigcap_{i\in I} R_i$ and $y \in S$ such that $y \le x$, then $y \in \bigcap_{i\in I} R_i$. Hence $\bigcap_{i\in I} R_i$ is an n-right ideal of S.

Lemma 3.5 Let S be an ordered LA-semigroup. The following statements are true:
(i) Every m-left ideal of S is an (m,n)-quasi-ideal of S.
(ii) Every n-right ideal of S is an (m,n)-quasi-ideal of S.

Proof. (i) Let A be an m-left ideal of S. Then $S^m A \subseteq A$ and $A \subseteq S$. It is obvious to see that A is an LA-subsemigroup of S. By considering $(S^m A] \cap (AS^n] \subseteq (S^m A] \subseteq (A] = A$. If $x \in A$ and $y \in S$ such that $y \le x$, then $y \in A$. Therefore A is an (m,n)-quasi-ideal of S.
(ii) Let B be an n-right ideal of S. Then $BS^n \subseteq B$ and $B \subseteq S$. Thus B is an LA-subsemigroup of S. By considering $(S^m B] \cap (BS^n] \subseteq (BS^n] \subseteq (B] = B$. If $x \in B$ and $y \in S$ such that $y \le x$, then $y \in B$. Therefore B is an (m,n)-quasi-ideal of S.

Theorem 3.6 Let S be an ordered LA-semigroup and let A be an m-left ideal and B be an n-right ideal of S. Then $A \cap B$ is an (m,n)-quasi-ideal of S.

Proof. By properties of A and B, we have $B^m A^n \subseteq BS^n \subseteq (BS^n]$ and $B^m A^n \subseteq S^m A \subseteq (S^m A]$. Hence $B^m A^n \subseteq (S^m A] \cap (BS^n] \subseteq (A] \cap (B] = A \cap B$, which prove that $A \cap B$ is non-empty. By Lemma 3.1, we get that $A \cap B$ is an LA-subsemigroup of S.

Now we show that $A \cap B$ is an (m,n)-quasi-ideal of S. Since A is an m-left ideal and B is an n-right ideal of S, we have $S^m A \subseteq A$ and $AS^n \subseteq A$. Moreover $(S^m(A \cap B)] \cap ((A \cap B)S^n] \subseteq (S^m A] \cap (BS^n] \subseteq (A] \cap (B] = A \cap B$. If $x \in A \cap B$ and $y \in S$ such that $y \le x$, then $y \in A \cap B$. That is $A \cap B$ is an (m,n)-quasi-ideal of S.

Definition 3.4 [12] A non-empty subset Q of an ordered LA-semigroup S is called idempotent if $A = (A^2]$.

Lemma 3.7 Every (m,n)-quasi-ideal Q of an ordered LA-semigroup S with left identity such that Q is an idempotent, is the intersection of some m-left ideal and some n-right ideal of S.

Proof. Let Q be an (m,n)-quasi-ideal of S. Setting $L=(Q\cup S^mQ]$ and $R=(Q\cup QS^n]$. First step we show that L is an LA-subsemigroup of S. Now

$LL=(Q\cup S^mQ](Q\cup S^mQ]\subseteq((Q\cup S^mQ)(Q\cup S^mQ)]=(QQ\cup Q(S^mQ)\cup(S^mQ)Q\cup(S^mQ)(S^mQ)]$
$=(QQ\cup S^m(QQ)\cup(QQ)S^m\cup(S^mS^m)(QQ)]=(QQ\cup S^m(QQ)\cup(QQ)(S^mS^m)\cup S^m(QQ)]$
$=(QQ\cup S^m(QQ)\cup(S^mS^m)(QQ))\cup S^m(QQ)]=(QQ\cup S^m(QQ)\cup S^m(QQ))\cup S^m(QQ)]$
$\subseteq(Q\cup S^mQ\cup S^mQ\cup S^mQ]\subseteq(Q\cup S^mQ]=L.$

Thus L is an LA-subsemigroup of S. Consequently,

$S^mL=S^m(Q\cup S^mQ]\subseteq(S^m](Q\cup S^mQ]\subseteq(S^m(Q\cup S^mQ)]=(S^mQ\cup S^m(S^mQ)]\subseteq(Q\cup S^m(S^mQ)]$
$=(Q\cup(S^mS^m)(S^mQ)]=(Q\cup(QS^m)(S^mS^m)]=(Q\cup(QS^m)S^m]=(Q\cup(S^mS^m)Q]=(Q\cup S^mQ]=L.$

It is to see that $x\in(Q\cup S^mQ]$ and $y\in S$ such that $y\le x$, then $y\in(Q\cup S^mQ]$.
Hence L is an m-left ideal of S. Similarly, R is an n-right ideal of S.
Since $Q\subseteq Q\cup(S^mQ]=(Q]\cup(S^mQ]=(Q\cup S^mQ]$ and $Q\subseteq Q\cup(QS^n]=(Q]\cup(QS^n]=(Q\cup QS^n]$.
We have $Q\subseteq(Q\cup S^mQ]\cap(Q\cup QS^n]$. Consider $(Q\cup S^mQ]\cap(Q\cup QS^n]$
$=((Q]\cup(S^mQ])\cap((Q]\cup(QS^n])=(Q]\cap((Q]\cup(QS^n])\cup(S^mQ]\cap((Q]\cup(QS^n])$
$=(((Q]\cap(Q])\cup((Q]\cap(QS^n]))\cup(((S^mQ]\cap(Q])\cup((S^mQ]\cap(QS^n]))=(Q]=Q$. Hence $Q=L\cap R$.

We further study the relation of (m,n)-quasi-ideals in regular ordered LA-semigroups.

Definition 3.5 [11] An element a of S is called a regular element of S if there exists some $x\in S$ such that $a\le(ax)a$ and S is called regular if every element of S is regular or equivalently, $A\subseteq((AS)A]$, $\forall A\subseteq S$ and $a\in((aS)a]$, $\forall a\in S$.

Now we will state and prove the intersection property of regular ordered LA-semigroups with (m,n)-quasi-ideals.

Lemma 3.8 Let S be a locally associative ordered LA-semigroup with left identity. If S is regular and $\varnothing\ne A\subseteq S$ such that A is an idempotent, then the following statements hold:

(i) $A\subseteq(S^mA]$ where $m\in\mathbb{Z}^+$.

(ii) $A\subseteq(AS^n]$ where $n\in\mathbb{Z}^+$.

Proof. (i) Let $P(m)$ be the statement $A\subseteq(S^mA]$, where $m\in\mathbb{Z}^+$, and let $x\in A$. Since S is regular, there exists $y\in S$ such that $x\le(xy)x$. Then $(xy)x\in SA$, and thus $x\in(SA]$. Therefore $A\subseteq(SA]$. Hence $P(1)$ holds true.

Let $P(k)$ holds true for all $k\in\mathbb{Z}^+$. We further show that $P(k+1)$ holds true. Then $A\subseteq(S^kA]$. Since S is a locally associative ordered LA-semigroup, we have $SA\subseteq S(S^mA]$ $\subseteq(S](S^mA]\subseteq(S(S^mA)]=((SS)(S^mA)]=((AS^m)(SS)]=((AS^m)S]=((SS^m)A]=(S^{m+1}A]$. Thus $A\subseteq(SA]\subseteq(S^{m+1}A]$. So $A\subseteq(S^{m+1}A]$. Therefore $P(k+1)$ is true. Hence $A\subseteq(S^mA]$ where $m\in\mathbb{Z}^+$.

(ii) Let $P(n)$ be the statement $A\subseteq(AS^n]$, where $n\in\mathbb{Z}^+$, and let $x\in A$. Since S is regular, there exists $y\in S$ such that $x\le(xy)x\in(AS)A=(AS)(AA]\subseteq((AS)(AA)]=((AA)(SA)]\subseteq$ $((AA)S]\subseteq((AA]S]=(AS]$. Thus $x\in(AS]$. Therefore $A\subseteq(AS]$. Hence $P(1)$ holds true.

Let $P(k)$ be true for all $k \in \mathbb{Z}^+$. Now we show that $P(k+1)$ is true. Then $A \subseteq (AS^k]$. Since S is a locally associative ordered LA-semigroup, we have
$AS \subseteq (AS^n]S \subseteq (AS^n](S] \subseteq ((AS^n)S] = (((AA]S^n)S] \subseteq (((AA)S^n)S] = ((SS^n)(AA)] =$
$((AA)(S^nS)] \subseteq ((AA](S^nS)] = (AS^{n+1}]$. Thus $A \subseteq (AS] \subseteq (AS^{n+1}]$. So $A \subseteq (AS^{n+1}]$. Therefore $P(k+1)$ is true. Hence $A \subseteq (AS^n]$ where $n \in \mathbb{Z}^+$.

Definition 3.6 A subsemigroup Q of an ordered LA-semigroup S has the (m,n) intersection property if Q is the intersection of an m-left ideal and an n-right ideal of S.

Lemma 3.9 Let S be a locally associative ordered LA-semigroup. Then every (m,n)-quasi-ideal Q of a regular ordered LA-semigroup of S with left identity such that Q is an idempotent has the (m,n) intersection property.

Proof. Let Q be an (m,n)-quasi-ideal of a regular ordered LA-semigroup S. By Lemma 3.8, we have $Q \subseteq (QS^n]$ and so $(Q \cup QS^n] = (Q] \cup (QS^n] = (QS^n]$. Therefore $(S^mQ] \cap (Q \cup QS^n]$ $= (S^mQ] \cap (QS^n] \subseteq Q$. Since $Q \subseteq Q \cup (S^mQ] = (Q \cup S^mQ]$ and $Q \subseteq Q \cup (QS^n] = (Q \cup QS^n]$, we have $Q \subseteq (Q \cup S^mQ] \cap (Q \cup QS^n]$. Now, $(Q \cup S^mQ] \cap (Q \cup QS^n] = (Q \cup (S^mQ]) \cap (Q \cup (QS^n]) =$ $(Q \cap (Q \cup (QS^n])) \cup ((S^mQ] \cap (Q \cup (QS^n])) = Q$. Hence $(Q \cup S^mQ] \cap (Q \cup QS^n] = Q$, which shows that Q has the intersection property.

Lemma 3.10 Let S be a locally associative ordered LA-semigroup with left identity and let S be a regular ordered LA-semigroup and let A be a non-empty subset of S such that A is an idempotent. Then A is an (m,n)-quasi-ideal of S if and only if it is the intersection of an m-left ideal and an n-right ideal.

Proof. $(\Rightarrow)$ Let A be an (m,n)-quasi-ideal of S. Then $(S^mA] \cap (AS^n] \subseteq A$. Next we can prove that $(S^mA]$ is an m-left ideal and $(AS^n]$ is an n-right ideal of S. We have that $(S^mA](S^mA] \subseteq ((S^mA)(S^mA)] = ((S^mS^m)(AA)] = (S^m(AA)] \subseteq (S^m(AA]] = (S^mA]$. Thus $(S^mA]$ is an LA-subsemigroup of S. We see that $S^m(S^mA] \subseteq (S^m](S^mA] \subseteq (S^m(S^mA)] = ((S^mS^m)(S^mA)]$ $= ((AS^m)(S^mS^m)] = ((AS^m)S^m] = ((S^mS^m)A] = (S^mA]$. And let $x \in (S^mA]$ and $y \in S$ such that $y \leq x$, then $y \in (S^mA]$. Therefore $(S^mA]$ is an m-left ideal of S. In a similar way, we have that $(AS^n](AS^n] \subseteq ((AS^n)(AS^n)] = ((AA)(S^nS^n)] = ((AA)S^n] \subseteq ((AA]S^n] = (AS^n]$. Thus $(AS^n]$ is an LA-subsemigroup of S. We see that $(AS^n]S^n \subseteq (AS^n](S^n] \subseteq ((AS^n)S^n] = (((AA]S^n)S^n] \subseteq$ $(((AA)S^n)S^n] = ((S^nS^n)(AA)] = (A((S^nS^n)A)] = (A((AS^n)S^n)] = ((AS^n)(AS^n)] = ((AA)(S^nS^n)]$ $\subseteq ((AA](S^nS^n)] = (AS^n]$. And if $x \in (AS^n]$ and $y \in S$ such that $y \leq x$, then $y \in (AS^n]$. Therefore $(AS^n]$ is an n-right ideal of S. By Lemma 3.8, we have $A \subseteq (S^mA]$ and $A \subseteq (AS^n]$. Then $A \subseteq (S^mA] \cap (AS^n]$. Hence $A = (S^mA] \cap (AS^n]$. Therefore A is the intersection of an m-left ideal and an n-right ideal.

$(\Leftarrow)$ Let A be an intersection of an m-left ideal and an n-right ideal. By Theorem 3.6, we get that A is an (m,n)-quasi-ideal of S.

4. Research Findings

The aforementioned content is entirely partial and provides no attempt to cover an ordered LA-semigroups. Thus this can be concluded that (m,n)-quasi-ideals in ordered LA-semigroups, was caused by the intersection of an m-left ideals and an n-right ideals of S. And if S was set to be locally associative ordered LA-semigroups then let S be regular ordered LA-semigroups. Then (m,n)-quasi-ideals of S caused by the intersection of an m-left ideals and an n-right ideals of S.

5. Discussion and research recommendations

We introduced the notion of (m,n)-quasi-ideals in ordered LA-semigroups as a generalization of the intersection of an m-left ideals and an n-right ideals of ordered LA-semigroups. We will prove that every (m,n)-quasi-ideal of a regular ordered LA-semigroup has the (m,n) intersection property. In continuity of this paper, we study (m,n) quasi-ideal of gamma ordered LA-semigroup.

6. References

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