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# $\Gamma$ - กึ่งกรุปอันดับที่บรรจุฐานสองด้าน

## On Ordered $\Gamma$ - Semigroups Containing Two-sided Bases

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**คำสำคัญ** : แกมมากึ่งกรุปอันดับ ; ฐานสองด้าน ; แกมมาไอดีล

## Abstract

The aim of this paper is to study the concept of ordered $\Gamma$-semigroups containing two-sided bases that are studied analogously to the concept of $\Gamma$-semigroups containing two-sided bases considered by T. Changpas and P. Kummoon in 2018. Moreover, we prove any ordered $\Gamma$-semigroups containing two-sided bases have the same cadinality.

**Keywords** : ordered $\Gamma$-semigroup ; two-sided bases ; $\Gamma$-Ideal

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**Introduction**

The notion of two-sided bases of semigroups has been introduced and studied by I.Fabrici. (Fabrici, 1975). Indeed, a non-empty subset $A$ of a semigroup $S$ is said to be a two-sided base of $S$ if $A$ satisfies the two following conditions :

(1) $S = A \cup SA \cup AS \cup SAS$;

(2) If $B$ is a subset of $A$ such that $S = B \cup SB \cup BS \cup SBS$, then $B = A$.

The concept of a $\Gamma$-semigroup has been introduced by M. K. Sen. (Sen, 1981). The concept of $\Gamma$-semigroups containing two-sided bases was firstly given by T. Changphas and P. Kummoon. (Thawhat & Pisit, 2018). Indeed, a non-empty subset $A$ of $S$ is called a two-sided base of $S$ if it satisfies the two following conditions:

(1) $S = A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S$;

(2) If $B$ is a subset of $A$ such that $S = B \cup S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S$, then $B = A$.

The main purpose of this paper is to introduce the concept and extend the result to an ordered $\Gamma$-semigroup containing two-sided bases. It will get the form of ordered $\Gamma$-semigroups containing two-sided bases is a non-empty subset $A$ of an ordered $\Gamma$-semigroup $S$ is called a two-sided base of $S$ if it satisfies the two following conditions:

(1) $S = (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S]$;

(2) If $B$ is a subset of $A$ such that $S = (B \cup S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S]$, then $B = A$.

We now recall some definitions and results used throughtout the paper.

**Definition 1.1.** (Thawhat & Pisit, 2018). Let $S$ and $\Gamma$ be any two non-empty sets. Then $S$ is called a $\Gamma$-semigroup if there exists a mapping from $S \times \Gamma \times S \to S$, written as $(a, \gamma, b) \mapsto a\gamma b$, satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

**Definition 1.2.** (Abdul *et al.*, 2017). Let $(S, \Gamma, \leq)$ be a $\Gamma$-semigroup. For $A$ and $B$ be two non-empty subsets of $S$, the set product $A\Gamma B$ is defined to be the set of all elements $a\gamma b$ in $S$ where $a \in A, b \in B$ and $\gamma \in \Gamma$. That is

$$A\Gamma B := \left\{ a\gamma b \,\middle|\, a \in A, b \in B, \gamma \in \Gamma \right\}.$$

Also we write $B\Gamma a$ instead of $B\Gamma\{a\}$, and we write $a\Gamma B$ instead of $\{a\}\Gamma B$, for $a \in S$.

**Definition 1.3.** (Niovi, 2017). An ordered $\Gamma$-semigroup is a $\Gamma$-semigroup $S$ together with a partial order relation $\leq$ on $S$ such that $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for all $a, b, c \in S$ and $\gamma \in \Gamma$.

(lampan, 2009). For an element $a$ of an ordered $\Gamma$-semigroup $S$, define $(a] := \left\{ t \in S \,\middle|\, t \leq a \right\}$ and for a subset $H$ of $S$, define $(H] = \bigcup_{h \in H} (h]$, that is $(H] = \left\{ t \in S \,\middle|\, t \leq h \text{ for some } h \in H \right\}$. Then the following holds true:

1. $H \subseteq (H] = ((H]]$;
2. For any subsets $A$ and $B$ of $S$ with $A \subseteq B$, we have $(A] \subseteq (B]$;
3. For any subsets $A$ and $B$ of $S$, we have $(A \cup B] = (A] \cup (B]$;
4. For any subsets $A$ and $B$ of $S$, we have $(A \cap B] \subseteq (A] \cap (B]$.

**Definition 1.4.** (Niovi, 2017). A non-empty subset $A$ of an ordered $\Gamma$-semigroup $(S, \Gamma, \leq)$ is called a $\Gamma$-subsemigroup of $S$ if $A\Gamma A \subseteq A$.

**Definition 1.5.** (Kwon and Lee, 1998). A non-empty subset $A$ of an ordered $\Gamma$-semigroup $(S, \Gamma, \leq)$ is called a left (resp. right) $\Gamma$-ideal of $S$ if it satisfies :

(1) $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$ ) ;

(2) if $a \in A$ and $b \leq a$ for $b \in S$ implies $b \in A$.

Both a left $\Gamma$-ideal and a right $\Gamma$-ideal of an ordered $\Gamma$-semigroup $S$ is called a $\Gamma$-ideal of $S$.

**Definition 1.6.** (Kostaq & Edmond, 2006). An $\Gamma$-ideal $A$ of an ordered $\Gamma$-semigroup $(S, \Gamma, \leq)$ is called proper if $A \neq S$. A proper $\Gamma$-ideal $A$ of $S$ is called maximal if for each $\Gamma$-ideal $T$ of $S$ such that $A \subseteq T$, we have $T = A$ or $T = S$, i.e., there is no $\Gamma$-ideal $T$ of $S$ such that $A \subset T \subset S$.

**Proposition 1.7.** (Kostaq & Edmond, 2006). Let $(S, \Gamma, \leq)$ be an ordered $\Gamma$-semigroup and $\{A_i \mid i \in I\}$ a family of $\Gamma$-ideals of $S$. If $\cap\{A_i \mid i \in I\} \neq \varnothing$, then the set $\cap\{A_i \mid i \in I\}$ is a $\Gamma$-ideal of $S$ and $\cup\{A_i \mid i \in I\}$ is also a $\Gamma$-ideal of $S$.

It is known (Niovi, 2017) that if denoted by $I(A)$, is the smallest $\Gamma$-ideal of $S$ containing $A$, and $I(A)$ is of the form $I(A) = (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S]$. In particular, for an element $a \in S$, we write $I(\{a\})$, $I(a)$ which is called the principal $\Gamma$-ideal of $S$ generated by $a$. Thus $I(a) = (a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]$. Note that for any $b \in S$, we have $(S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S]$ is a $\Gamma$-ideal of $S$. Finally, if $A$ and $B$ are two $\Gamma$-ideals of $S$, then the union $A \cup B$ is a $\Gamma$-ideal of $S$.

## Methods

We begin this section with the definitions of two-sided bases of ordered $\Gamma$-semigroups.

**Definition 2.1.** (Abul *et al.*, 2017). Let $(S, \Gamma, \leq)$ be an ordered $\Gamma$-semigroup. A non-empty subset $A$ of $S$ is called a two-sided base of $S$ if it satisfies the two following conditions :

(1) $S = (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S]$;

(2) If $B$ is a subset of $A$ such that $S = (B \cup S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S]$, then $B = A$.

We now provide some examples.

**Example 2.2.** (Chinnadurai & Arulmozhi, 2018). Let $S=\{a,b,c,d\}$ and $\Gamma=\{\alpha,\beta\}$ where $\alpha,\beta$ are define on $S$ with the following Cayley tables:

| $\alpha$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $c$ | $d$ |
| $c$ | $a$ | $c$ | $c$ | $c$ |
| $d$ | $a$ | $c$ | $c$ | $c$ |

| $\beta$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $c$ | $d$ |
| $c$ | $a$ | $c$ | $c$ | $c$ |
| $d$ | $a$ | $b$ | $c$ | $d$ |

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(b,b),(b,c),(b,d),(c,c),(d,c),(d,d)\}$$

In (Chinnadurai & Arulmozhi, 2018). $(S,\Gamma,\leq)$ is an ordered $\Gamma$-semigroup. It is easy to see that the two-sided bases of $S$ are $\{b\}$ and $\{d\}$. But $\{b,d\}$ is not a two-sided base.

**Example 2.3.** (Subrahmanyeswara *et al.*, 2012). Let $S=\{a,b,c,d\}$ and $\Gamma=\{\alpha\}$ where $\alpha$ is defined on $S$ with the following Cayley tables:

| $\alpha$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $b$ | $b$ | $d$ | $d$ |
| $b$ | $b$ | $b$ | $d$ | $d$ |
| $c$ | $d$ | $d$ | $c$ | $d$ |
| $d$ | $d$ | $d$ | $d$ | $d$ |

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(a,b),(d,b),(d,c)\}$$

In (Subrahmanyeswara *et al.*, 2012). $(S,\Gamma,\leq)$ is an ordered $\Gamma$-semigroup. It is easy to see that the two-sided bases of $S$ is $\{a,c\}$. But $\{b\}$ and $\{d\}$ are not a two-sided bases.

In Example 2.2. and Example 2.3., it is observed that two-sided bases of $S$ have same cardinality. This leads to prove in Theorem 3.4.

Hereafter, for any ordered $\Gamma$-semigroup $(S,\Gamma,\leq)$, we shall use the quasi-ordering which is defined as follows.

**Definition 2.4.** Let $(S,\Gamma,\leq)$ be an ordered $\Gamma$-semigroup. We define a quasi-ordering $\preceq$ on $S$ for any $a,b\in S$,

$$a \preceq_I b \Leftrightarrow I(a)\subseteq I(b).$$

We write $a \prec_I b$ if $a \preceq_I b$ but $a\neq b$. It is clear that, for any $a,b$ in $S$, $a\leq b$ implies $a \preceq_I b$.

**Lemma 2.5.** Let $A$ be a two-sided base of an ordered $\Gamma$-semigroup $(S,\Gamma,\leq)$, and $a,b\in A$. If $a\in (S\Gamma b\cup b\Gamma S\cup S\Gamma b\Gamma S]$, then $a=b$.

**Proof.** Assume that $a \in (S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S]$, and suppose that $a \neq b$. Let $B = A \setminus \{a\}$. Since $a \neq b$, $b \in B$. To show that $I(A) \subseteq I(B)$, it suffices to show that $A \subseteq I(B)$. Let $x \in A$. There are two cases to consider. If $x \neq a$, then $x \in B$, and so $x \in I(B)$. If $x \neq a$, then by assumption we have $x = a \in (S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S] \subseteq I(b) \subseteq I(B)$. So we have $I(A) \subseteq I(B)$. Thus $S = I(A) \subseteq I(B) \subseteq S$. This is a contradiction. Hence $a = b$.

**Results**

In this part, the algebraic structure of an ordered $\Gamma$-semigroup containing two-sided bases will be presented.

**Theorem 3.1.** A non-empty subset $A$ of an ordered $\Gamma$-semigroup $(S, \Gamma, \leq)$ is a two-sided base of $S$ if and only if $A$ satisfies the two following conditions:

(1) For any $x \in S$ there exists $a \in A$ such that $x \preceq_I a$;

(2) For any $a, b \in A$, if $a \neq b$, then neither $a \preceq_I b$ nor $b \preceq_I a$.

**Proof.** Assume first that $A$ is a two-sided base of $S$. Then $I(A) = S$. Let $x \in S$ Then $x \in I(A) = \bigcup_{a \in A} I(a)$, and so $x \in I(a)$ for some $a \in A$. This implies $I(x) \subseteq I(a)$. Hence $x \preceq_I a$. Hence condition (1) is true. Let $a, b \in A$ such that $a \neq b$ Suppose that $a \preceq_I b$. We set $B = A \setminus \{a\}$. Then $b \in B$. Let $x \in S$. By (1), there exists $c \in A$ such that $x \preceq_I c$. There are two cases to consider. If $c \neq a$, then $c \in B$, thus $I(x) \subseteq I(c) \subseteq I(B)$. Hence $S = I(B)$. This is a contradiction. If $c \neq a$, then $x \preceq_I b$ hence $x \in I(B)$ since $b \in B$. We have $S = I(B)$. This is a contradiction. The case $b \preceq_I a$ is proved similarly. Hence condition (2) is true.

Conversely, assume that the condition (1) and (2) hold. We will show that $A$ is a two-sided base of $S$. To show that $S = I(A)$. Let $x \in S$. By (1), then there exists $a \in A$ such that $I(x) \subseteq I(a)$. Then $x \in I(x) \subseteq I(a) \subseteq I(A)$. Thus $S \subseteq I(A)$, and $S = I(A)$. Next it remains to show that $A$ is a minimal subset of $S$ with the property: $S = I(A)$. Suppose that $S = I(B)$ for some $B \subset A$. Since $B \subset A$, there exists $a \in A \setminus B$. So $a \notin (B]$. Since $a \in A \subseteq S = I(B)$ and $a \notin (B]$, it follows that $a \in (S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S]$. Since $a \in (S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S]$, we have $a \leq y$ for some $y \in S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S$. There are three cases to consider:

**Case 1:** $y \in B\Gamma S$. Then $y = b_1\gamma s$ for some $b_1 \in B, \gamma \in \Gamma$ and $s \in S$. Since $a \leq y$ and $y \in b_1 \cup S\Gamma b_1 \cup b_1\Gamma S \cup S\Gamma b_1\Gamma S$, we have $a \in (b_1 \cup S\Gamma b_1 \cup b_1\Gamma S \cup S\Gamma b_1\Gamma S]$. It follows that $I(a) \subseteq I(b_1)$. Hence, $a \preceq_I b_1$. This is a contradiction.

**Case 2:** $y \in S\Gamma B$. Then $y = s\gamma b_2$ for some $b_2 \in B, \gamma \in \Gamma$ and $s \in S$. Since $a \leq y$ and $y \in b_2 \cup S\Gamma b_2 \cup b_2\Gamma S \cup S\Gamma b_2\Gamma S$, we have $a \in (b_2 \cup S\Gamma b_2 \cup b_2\Gamma S \cup S\Gamma b_2\Gamma S]$. It follows that $I(a) \subseteq I(b_2)$. Hence, $a \preceq_I b_2$. This is a contradiction.

Case 3: $y \in S\Gamma B\Gamma S$. Then $y = s_1\gamma_1 b_3\gamma_2 s_2$ for some $b_3 \in B, \gamma_1, \gamma_2 \in \Gamma$ and $s \in S$. Since $a \le y$ and $y \in b_3 \cup S\Gamma b_3 \cup b_3\Gamma S \cup S\Gamma b_3\Gamma S$, we have $a \in (b_3 \cup S\Gamma b_3 \cup b_3\Gamma S \cup S\Gamma b_3\Gamma S]$. It follows that $I(a) \subseteq I(b_3)$. Hence $a \preceq_I b_3$. This is a contradiction.

Therefore $A$ is a two-sided base of $S$ as required, and the proof is completed.

**Theorem 3.2.** Let $A$ be a two-sided base of an ordered $\Gamma$-semigroup $(S, \Gamma, \le)$ such that $I(a) = I(b)$ for some $a \in A$ and $b \in S$. If $a \ne b$, then $S$ contains at least two two-sided bases.

**Proof.** Assume that $a \neq b$. Suppose that $b \in A$. Since $a \preceq_I b$ and $a \ne b$, it follows that $a \in (S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S]$. By Lemma 2.5., we obtain $a = b$. This is a contradiction. Thus $b \in S \setminus A$. Let $B := (A \setminus \{a\}) \cup \{b\}$. Since $b \in B$, we have $b \notin A$, and $B \not\subseteq A$. Hence $A \ne B$. We will show that $B$ is a two-sided base of $S$. To show that $B$ satisfies (1) in Theorem 3.1., let $x \in S$. Since $A$ is a two-sided base of $S$, there exists $c \in A$ such that $x \preceq_I c$. If $c \neq a$, then $c \in B$. If $c = a$, then $x \preceq_I a$. Since $a \preceq_I b$, $x \preceq_I a \preceq_I b$, we have $x \preceq_I b$. To show that $B$ satisfies (2) in Theorem 3.1., let $c_1, c_2 \in B$ be such that $c_1 \neq c_2$. We will show that neither $c_1 \preceq_I c_2$ nor $c_2 \preceq_I c_1$. Since $c_1 \in B$ and $c_2 \in B$, we have $c_1 \in A \setminus \{a\}$ or $c_1 = b$ and $c_2 \in A \setminus \{a\}$ or $c_2 = b$. There are four cases to consider:

Case 1: $c_1 \in A \setminus \{a\}$ and $c_2 \in A \setminus \{a\}$. This implies neither $c_1 \preceq_I c_2$ nor $c_2 \preceq_I c_1$.

Case 2: $c_1 \in A \setminus \{a\}$ and $c_2 = b$. If $c_1 \preceq_I c_2$, then $c_1 \preceq_I b$. Since $b \preceq_I a$, $c_1 \preceq_I b \preceq_I a$. Thus $c_1 \preceq_I a$, a contradiction. If $c_2 \preceq_I c_1$, then $b \preceq_I c_1$. Since $a \preceq_I b$, $a \preceq_I b \preceq_I c_1$. So $a \preceq_I c_1$, a contradiction.

Case 3: $c_2 \in A \setminus \{a\}$ and $c_1 = b$ . If $c_1 \preceq_I c_2$, then $b \preceq_I c_2$. Since $a \preceq_I b$, $a \preceq_I b \preceq_I c_2$. Hence $a \preceq_I c_2$, a contradiction. If $c_2 \preceq_I c_1$, then $c_2 \preceq_I b$ . Since $b \preceq_I a$, $c_2 \preceq_I b \preceq_I a$. Thus $c_2 \preceq_I a$, a contradiction.

Case 4: $c_1 = b$ and $c_2 = b$. This is impossible.

Thus $B$ satisfies (1) and (2) in Theorem 3.1. Therefore $B$ is a two-sided base of $S$.

**Corollary 3.3.** Let $A$ be a two-sided base of an ordered $\Gamma$-semigroup $(S, \Gamma, \le)$, and let $a \in A$. If $I(x) = I(a)$ for some $x \in S$, $x \ne a$, then $x$ belongs to a two-sided base of $S$, which is different from $A$.

**Theorem 3.4.** Let $A$ and $B$ be any two-sided bases of an ordered $\Gamma$-semigroup $(S, \Gamma, \le)$. Then $A$ and $B$ have the same cardinality.

**Proof.** Let $a \in A$. Since $B$ is a two-sided base of $S$, by Theorem 3.1.(1), there exists an element $b \in B$ such that $a \preceq_I b$. Since $A$ is a two-sided base of $S$, by Theorem 3.1.(1), there exists $a^* \in A$ such that $b \preceq_I a^*$. So $a \preceq_I b \preceq_I a^*$, i.e., $a \preceq_I a^*$. By Theorem 3.1.(2), $a = a^*$. Hence $I(a) = I(b)$. Define a mapping

$$\varphi : A \to B \text{ by } \varphi(a) = b \text{ for all } a \in A.$$

To show that $\varphi$ is well-defined, let $a_1, a_2 \in A$ be such that $a_1 = a_2$, $\varphi(a_1) = b_1$ and $\varphi(a_2) = b_2$ for some

$b_1, b_2 \in B$. Then $I(a_1) = I(b_1)$ and $I(a_2) = I(b_2)$. Since $a_1 = a_2$, $I(a_1) = I(a_2)$. Hence $I(a_1) = I(a_2) = I(b_1) = I(b_2)$, i.e., $b_1 \preceq_I b_2$ and $b_2 \preceq_I b_1$. By Theorem 3.1.(2), $b_1 = b_2$. Thus $\varphi(a_1) = \varphi(a_2)$. Therefore, $\varphi$ is well-defined. We will show that $\varphi$ is one-one. Let $a_1, a_2 \in A$ be such that $\varphi(a_1) = \varphi(a_2)$. Since $\varphi(a_1) = \varphi(a_2)$, $\varphi(a_1) = \varphi(a_2) = b$ for some $b \in B$. So $I(a_2) = I(a_1) = I(b)$. Since $I(a_2) = I(a_1)$, $a_1 \preceq_I a_2$ and $a_2 \preceq_I a_1$. This implies $a_1 = a_2$. Therefore, $\varphi$ is one-one. We will show that $\varphi$ is onto. Let $b \in B$. Since $A$ is a two-sided base of $S$, by Theorem 3.1.(1), there exists an element $a \in A$ such that $b \preceq_I a$. Since $B$ is a two-sided base of $S$, by Theorem 3.1.(1), there exists an element $b^* \in B$ such that $a \preceq_I b^*$. So $b \preceq_I a \preceq_I b^*$, i.e., $b \preceq_I b^*$. This implies $b = b^*$. Hence $I(a) = I(b)$. Thus $\varphi(a) = b$. Therefore, $\varphi$ is onto. This completes the proof.

If a two-sided base $A$ of an ordered $\Gamma$-semigroup $(S, \Gamma, \leq)$ is a $\Gamma$-ideal of $S$, then $S = (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S] \subseteq (A] = A$. Hence $S = A$. The converse statement is obvious. Then we conclude that.

**Remark 3.5.** It is observed that a two-sided base $A$ of an ordered $\Gamma$-semigroup $(S, \Gamma, \leq)$ is a two-sided $\Gamma$-ideal of $S$ if and only if $A = S$.

In Example 2.2., it is easy to see that $\{d\}$ is a two-sided base of $S$, but it is not a $\Gamma$-subsemigroup of $S$ This shows that a two-sided base of an ordered $\Gamma$-semigroup need not to be a $\Gamma$-subsemigroup in (Niovi, 2018). A non-empty subset $A$ of $S$ is called an idempotent if $A = (A\Gamma A]$ or $a = a\gamma a$ for all $a \in A$ and $\gamma \in \Gamma$. The following theorem gives necessary and sufficient conditions of a two-sided base of $S$ to be a $\Gamma$-subsemigroup $S$.

**Theorem 3.6.** A two-sided base $A$ of an ordered $\Gamma$-semigroup $(S, \Gamma, \leq)$ is a $\Gamma$-subsemigroup if and only if $A = \{a\}$ with $a\gamma a = a$ for all $\gamma \in \Gamma$.

**Proof.** Assume that $A$ is a $\Gamma$-subsemigroup of $S$. Let $a, b \in A$ and $\gamma \in \Gamma$. Since $A$ is a $\Gamma$-subsemigroup of $S$, $a\gamma b \in A$. Setting $a\gamma b = c$ ; thus $c \in S\Gamma b \subseteq S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma B \subseteq (S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma B]$. By Lemma 2.5., $c = b$. So $a\gamma b = b$. Similarly, $c \in a\Gamma S \subseteq S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S \subseteq (S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]$. By Lemma 2.5., $c = a$. So $a\gamma b = a$. We have $a = b$. Therefore, $A = \{a\}$ with $a\gamma a = a$ for all $a \in A$ and $\gamma \in \Gamma$. The converse statement is clear.

**Notation.** The union of all two-sided bases of an ordered $\Gamma$-semigroup $(S, \Gamma, \leq)$ is denoted by $R$.

**Theorem 3.7.** Let $(S, \Gamma, \leq)$ be an ordered $\Gamma$-semigroup. Then $S \setminus R$ is either empty set or a $\Gamma$-ideal of $S$.

**Proof.** Assume that $S \setminus R \neq \varnothing$. We will show that $S \setminus R$ is a $\Gamma$-ideal of $S$. Let $a \in S \setminus R$, $x \in S$ and $\gamma \in \Gamma$. To show that $x\gamma a \in S \setminus R$ and $a\gamma x \in S \setminus R$. Suppose that $x\gamma a \not\subseteq S \setminus R$. Then $x\gamma a \in R$. Hence $x\gamma a \in A$ for some a two-sided base $A$ of $S$. We set $b = x\gamma a$ for some $b \in A$. Then $b \in S\Gamma a$. By $b \in S\Gamma a \subseteq a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S \subseteq (a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S] = I(a)$, it follow that $I(b) \subseteq I(a)$. Next we will show that $I(b) \subset I(a)$. Suppose that $I(b) = I(a)$. Since $a \in S \setminus R$ and $b \in A$, $a \neq b$. Since $I(b) = I(a)$ and Corollary 3.3, we conclude that $a \in R$. This is a contradiction. Thus $I(b) \subset I(a)$, i.e., $b \prec_I a$. Since $A$ is a two-sided base of $S$ and $a \in S \setminus R$, by Theorem 3.1.(1), there exists $d \in A$ such that $a \preceq_I d$. Since $b \prec_I a \preceq_I d$, $b \preceq_I d$. This is a contradiction to the condition (2) of Theorem 3.1., so we have $x\gamma a \in S \setminus R$. Similarly, to show that $a\gamma x \in S \setminus R$. Suppose that $a\gamma x \in R$, then $a\gamma x \in A$ for some a two-sided base $A$ of $S$. Let $a\gamma x = c$ for some $c \in A$. Then $c \in a\Gamma S$. By $c \in a\Gamma S \subseteq a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S \subseteq (a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S] = I(a)$, it follow that $I(c) \subseteq I(a)$. Next, we will show that $I(c) \subset I(a)$. Suppose that $I(c) = I(a)$. Since $a \in S \setminus R$ and $c \in A$, $a \neq c$. Since $I(c) = I(a)$ and Corollary 3.3., we conclude that $a \in R$. This is a contradiction. Thus $I(c) \subset I(a)$, i.e., $c \prec_I a$. Since $A$ is a two-sided base of $S$ and $a \in S \setminus R$, by Theorem 3.1.(1), there exists $e \in A$ such that $a \preceq_I e$. Since $c \prec_I a \preceq_I e$, $c \preceq_I e$. This is a contradiction to the condition (2) of Theorem 3.1., so we have $a\gamma x \in S \setminus R$. Let $x \in S \setminus R$, $y \in S$ such that $y \leq x$. Next we will show that $y \in S \setminus R$. Suppose that $y \in R$, then $y \in A$ for some a two-sided base $A$ of $S$. Since $A$ is a two-sided bases of $S$, by Theorem 3.1.(1) there exists an element $z \in A$ such that $x \preceq_I z$. Since $y \leq x$, $y \preceq_I x$ and $x \preceq_I z$. So we have $y \preceq_I z$. This is a contradiction. Therefore $y \notin R$ then $y \in S \setminus R$. Hence $S \setminus R$ is a $\Gamma$-ideal of $S$.

**Notation.** Let $M^*$ be a proper $\Gamma$-ideal of an ordered $\Gamma$-semigroup $(S, \Gamma, \leq)$ containing every proper $\Gamma$-ideal of $S$.

**Theorem 3.8.** Let $(S, \Gamma, \leq)$ be an ordered $\Gamma$-semigroup and $\varnothing \neq R \subset S$. The following statements are equivalent:

(1) $S \setminus R$ is a maximal proper $\Gamma$-ideal of $S$;

(2) For every element $a \in R$, $R \subseteq I(a)$;

(3) $S \setminus R = M^*$;

(4) Every two-sided base of $S$ is a one-element base.

**Proof.** (1) $\Leftrightarrow$ (2). Assume that $S \setminus R$ is a maximal proper $\Gamma$-ideal of $S$. Let $a \in R$. Suppose that $R \not\subseteq I(a)$. Since $R \not\subseteq I(a)$, there exists $x \in R$ such that $x \notin I(a)$. Hence $x \notin S \setminus R$. Since $x \notin I(a)$, $x \notin S \setminus R$ and $x \in S$, we have $(S \setminus R) \cup I(a) \subset S$. Thus $(S \setminus R) \cup I(a)$ is a proper $\Gamma$-ideal of $S$. Hence $S \setminus R \subset (S \setminus R) \cup I(a)$. This contradicts to the maximality of $S \setminus R$.

Conversely, assume that for every element $a \in R, R \subseteq I(a)$. We will show that $S \setminus R$ is a maximal proper $\Gamma$-ideal of $S$. Since $a \in R$, $a \notin S \setminus R$. Hence $S \setminus R \subset S$. Since $R \subset S, S \setminus R \neq \varnothing$. By Theorem 3.7., $S \setminus R$ is a proper $\Gamma$-ideal of $S$. Suppose that $M$ is a proper $\Gamma$-ideal of $S$ such that $S \setminus R \subset M \subset S$. Since $S \setminus R \subset M$, there exists $x \in M$ such that $x \notin S \setminus R$, i.e., $x \in R$. Then $x \in M \cap R$. So $M \cap R \neq \varnothing$. Let $c \in M \cap R$. Then $c \in M$ and $c \in R$. Since $c \in M$, we have $S\Gamma c \subseteq S\Gamma M \subseteq M$, $c\Gamma S \subseteq M\Gamma S \subseteq M$ and $S\Gamma c\Gamma S \subseteq S\Gamma M\Gamma S \subseteq M$. Then $I(c) = (c \cup S\Gamma c \cup c\Gamma S \cup S\Gamma c\Gamma S] \subseteq M$. Since $c \in R$, by assumption we have $R \subseteq I(c)$. Hence

$S=(S\setminus R)\cup\ R\subseteq(S\setminus R)\cup I(c)\subseteq M\subset S$. Thus $M=S$. This is a contradiction. Therefore $S\setminus R$ is a maximal proper $\Gamma$-ideal of $S$.

(3) $\Leftrightarrow$ (4). Assume that $S\setminus R=M^*$. Since $S\setminus R=M^*$, $S\setminus R$ is a maximal proper $\Gamma$-ideal of $S$. By (1) $\Leftrightarrow$ (2), for every $a\in R, R\subseteq I(a)$. Firstly, we will show that for every $a\in R, S\setminus R\subseteq I(a)$. Suppose that $S\setminus R\not\subseteq I(a)$ for some $a\in R$. Then $I(a)\neq S$. Hence $I(a)$ is a proper $\Gamma$-ideal of $S$. Thus $I(a)\subseteq M^*=S\setminus R$. Then $I(a)\subseteq S\setminus R$. Since $a\in I(a), a\in S\setminus R$, i.e., $a\notin R$. This is a contradiction. Thus $S\setminus R\subseteq I(a)$ for every $a\in R$. Since $S\setminus R\subseteq I(a)$ and $R\subseteq I(a)$ for every $a\in R$, it follows that $S=(S\setminus R)\cup R\subseteq I(a)\cup I(a)=I(a)\subseteq S$. So $S=I(a)$ for every $a\in R$. Therefore, $\{a\}$ is a two-sided base of $S$. Let $A$ be a two-sided base of $S$. We will show that $a=b$ for all $a,b\in A$. Suppose that there exists $a,b\in A$ such that $a\neq b$. Since $A$ is a two-sided base of $S$, $A\subseteq R$. This is, $a\in R$. So $S=I(a)$. Since $b\in S=I(a)$ and $b\neq a$, $b\in(S\Gamma a\cup a\Gamma S\cup S\Gamma a\Gamma S]$. By Lemma 2.5., $a=b$. This is a contradiction. Therefore, every two-sided base of $S$ is an one element base.

Conversely, assume that every two-sided base of $S$ is an one element base. Then $S=I(a)$ for all $a\in R$. We will show that $S\setminus R=M^*$. The statement that $S\setminus R$ is a maximal proper $\Gamma$-ideal of $S$ follows from the proof (1) $\Leftrightarrow$ (2). Let $M$ be a $\Gamma$-ideal of $S$ such that $M$ is not contained in $S\setminus R$. Then $R\cap M\neq\varnothing$. Let $a\in R\cap M$. Hence $a\in R$ and $a\in M$. So $S\Gamma a\subseteq S\Gamma M\subseteq M$, $a\Gamma S\subseteq M\Gamma S\subseteq M$ and $S\Gamma a\Gamma S\subseteq S\Gamma M\Gamma S$. So we have $I(a)=(a\cup S\Gamma a\cup a\Gamma S\cup S\Gamma a\Gamma S]\subseteq M$. Hence $S=I(a)\subseteq M\subseteq S$. Thus $M=S$. Therefore $S\setminus R=M^*$

(1) $\Leftrightarrow$ (3). Assume that $S\setminus R$ is a maximal proper $\Gamma$-ideal of $S$. We will show that $S\setminus R=M^*$. Since $S\setminus R$ is a proper $\Gamma$-ideal of $S$, $S\setminus R\subseteq M^*\subset S$. By assumption, $S\setminus R=M^*$ or $S=M^*$. Since $S\neq M^*$, so we have $S\setminus R=M^*$. The converse statement is obvious.

## Discussion

In this research, we investigated the notion of ordered $\Gamma$-semigroup containing two-sided bases. We proved that a non-empty subset $A$ of an ordered $\Gamma$-semigroup $(S,\Gamma,\leq)$ is a two-sided base of $S$ if and only if $A$ satisfies the two following conditions (1) for any $x\in S$ there exists $a\in A$ such that $x\preceq_I a$; (2) for any $a,b\in A$, if $a\neq b$, then neither $a\preceq_I b$ nor $b\preceq_I a$. Also, we showed that if $A$ and $B$ be any two-sided bases of an ordered $\Gamma$-semigroup $(S,\Gamma,\leq)$. Then $A$ and $B$ have the same cardinality.

## Conclusions

In this research, we have resulted in ordered $\Gamma$-semigroup that is analogously in $\Gamma$-semigroup considered by T. Changpas and P. Kummoon in 2018.

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