Original Article

On interior bases of a semigroup

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Abstract

The main purpose of this paper is to introduce the concept of interior bases of a semigroup. In addition, we give a characterization when a non-empty subset of a semigroup is an interior base of a semigroup and give necessary and sufficient conditions of an interior base of a semigroup to be a subsemigroup.

Keywords: semigroup, interior ideal, interior base, quasi-order

1. Introduction and Preliminaries

The notion of interior ideals of a semigroup has been introduced by Lajos (1976). Muhiuddin (2019) applied the cubic set theory to interior ideals of a semigroup. Muhiuddin and Mahboob (2020) introduced and studied int-soft interior ideals over the soft sets in ordered semigroups. Muhiuddin studied the concept of different types of ideals in semigroups, see (Muhiuddin, 2018; Muhiuddin, Mahboob, & Mohammad Khan, 2019). Based on the notion of interior ideals of a semigroup generated by a non-empty subset of a semigroup. The notion of one-sided bases of a semigroup was first introduced by Tamura (1955). Later, Fabrici (1972) studied the structure of a semigroup containing one-sided bases. After that, the concept of two-sided bases of a semigroup was studied by Fabrici (1975). Changphas and Summaprab (2014) introduced the concept of two-sided bases of an ordered semigroup. Recently, Kumnun and Changphas (2017) introduced the concept of bi-bases of a semigroup. The main purpose of this paper is to introduce the concept which is called interior bases of a semigroup. Also, we give a characterization when a non-empty subset of a semigroup is an interior base of the semigroup. Finally, we give necessary and sufficient conditions of an interior base of a semigroup to be a subsemigroup.

A semigroup is a pair $(S, \cdot)$ in which $S$ is a non-empty set and $\cdot$ is a binary associative operation on $S$, i.e., the equation $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ holds for all $x, y, z \in S$.

Throughout this paper, unless stated otherwise, we write the semigroup operation as multiplication and we mostly omit it typographically, i.e., we write $S$ instead of $(S, \cdot)$, $xy$ instead of $x \cdot y$, $x(yz)$ instead of $x \cdot (y \cdot z)$ and so on.

For $A$ and $B$ are non-empty subsets of a semigroup $S$, we define the set product $AB$ of $A$ and $B$, by

$$AB = \{ab \mid a \in A, b \in B\}.$$
For $a \in S$, we write $B_0$ for $B(a)$, and similarly for $aB$.

**Definition 1.1.** (Lajos, 1976) A non-empty subset $A$ of a semigroup $S$ is called a subsemigroup of $S$ if $AA \subseteq A$.

**Definition 1.2.** (Lajos, 1976) A subsemigroup $A$ of a semigroup $S$ is called an interior ideal of $S$ if $SAS \subseteq A$.

**Lemma 1.3.** Let $S$ be a semigroup and $A_i$ be a subsemigroup of $S$ for all $i \in I$. If $\bigcap_{i \in I} A_i \neq \emptyset$, then $\bigcap_{i \in I} A_i$ is a subsemigroup of $S$.

**Proof.** Assume that $\bigcap_{i \in I} A_i \neq \emptyset$. Let $a, b \in \bigcap_{i \in I} A_i$. Then $a, b \in A_i$ for all $i \in I$. Since $A_i$ is a subsemigroup of $S$ for all $i \in I$, so $ab \in A_i$ for all $i \in I$. Thus $ab \in \bigcap_{i \in I} A_i$. Therefore, $\bigcap_{i \in I} A_i$ is a subsemigroup of $S$.

**Lemma 1.4.** Let $S$ be a semigroup and $A_i$ be an interior ideal of $S$ for all $i \in I$. If $\bigcap_{i \in I} A_i \neq \emptyset$, then $\bigcap_{i \in I} A_i$ is an interior ideal of $S$.

**Proof.** Assume that $\bigcap_{i \in I} A_i \neq \emptyset$. By Lemma 1.3, $\bigcap_{i \in I} A_i$ is a subsemigroup of $S$. Next, we will show that $S(\bigcap_{i \in I} A_i)S \subseteq \bigcap_{i \in I} A_i$.

Let $x \in S(\bigcap_{i \in I} A_i)S$. Then $x = s_i a s_2$ for some $s_i, s_2 \in S$ and $a \in \bigcap_{i \in I} A_i$. Since $a \in \bigcap_{i \in I} A_i$, we have $a \in A_i$ for all $i \in I$, where $A_i$ is an interior ideal of $S$ for all $i \in I$. So we have $x = s_i a s_2 \in S(A_i)S \subseteq A_i$ for all $i \in I$. Thus $x \in \bigcap_{i \in I} A_i$.

Therefore, $\bigcap_{i \in I} A_i$ is an interior ideal of $S$.

**Definition 1.5.** Let $S$ be a semigroup and let $A$ be a non-empty subset of $S$. Then, the intersection of all interior ideals of $S$ containing $A$ is the smallest interior ideal of $S$ generated by $A$, denoted by $(A)_i$.

**Lemma 1.6.** Let $S$ be a semigroup and let $A$ be a non-empty subset of $S$. Then,

$$(A)_i = A \cup AA \cup SAS.$$ 

**Proof.** Let $B = A \cup AA \cup SAS$. Consider, 

$BB = (A \cup AA \cup SAS)(A \cup AA \cup SAS)$

$= AA \cup AAA \cup SAS \cup AAA \cup AAAA \cup AASAS \cup (SAS)A \cup SASAA \cup SASSAS$

$\subseteq AA \cup SAS \cup SAS \cup SAS \cup SAS \cup SAS \cup SAS \cup SAS \cup SAS \cup SAS$

$= AA \cup SAS \subseteq B.$

Thus $B$ is a subsemigroup of $S$. Next, consider 

$SBS = S(A \cup AA \cup SAS)S$

$= (SA \cup SAS \cup SSAS)S$

$= SAS \cup SAS \cup SSASS$

$\subseteq SAS \cup SAS \cup SAS = SAS \subseteq B.$

Thus $SBS \subseteq B$. Hence, $B$ is an interior ideal of $S$ containing $A$. Finally, let $C$ be an interior ideal of $S$ containing $A$. Clearly, $A \subseteq C$. Since $C$ is a subsemigroup of $S$, we have $AA \subseteq CC \subseteq C$. Since $C$ is an interior ideal of $S$, we have $SAS \subseteq SCS \subseteq C$. Thus $B = A \cup AA \cup SAS \subseteq C$. Hence, $B$ is the smallest interior ideal of $S$ containing $A$.

2. Main Results

In this part, the definition of interior bases of a semigroup and the algebraic structure of a semigroup containing interior bases will be presented.
**Definition 2.1.** Let $S$ be a semigroup. A non-empty subset $A$ of $S$ is called an interior base of $S$ if it satisfies the following two conditions:

1. $S = A \cup AA \cup SAS$, i.e., $S = (A)_I$;
2. if $B$ is a subset of $A$ such that $S = (B)_I$, then $B = A$.

**Example 2.2.** (Bussaban & Changhas, 2016) Let $S = \{a,b,c,d,f\}$ be a semigroup with the binary operation defined by:

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The interior bases of $S$ are $\{b\}$, $\{c\}$, $\{d\}$, and $\{f\}$.

**Example 2.3.** (Yaqoob, Aslam, & Chinram, 2012) Let $S = \{0,1,2,3\}$ be a semigroup with the binary operation defined by:

<table>
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The interior base of $S$ is $\{0,1\}$. But $\{0\}$ and $\{1\}$ are not interior bases of $S$.

First, we have the following useful lemma.

**Lemma 2.4.** Let $A$ be an interior base of a semigroup $S$, and let $a,b \in A$. If $a \in bb \cup ShS$, then $a = b$.

**Proof.** Assume that $a \in bb \cup ShS$, and suppose that $a \neq b$. Let $B = A \setminus \{a\}$. Then $B \subseteq A$. Since $a \neq b$, we have $b \in B$.

We will show that $(B)_I = S$. Clearly, $(B)_I \subseteq S$. Next, let $x \in S$. Then, by $(A)_I = S$, we have $x \in A \cup AA \cup SAS$. There are three cases to consider:

**Case 1:** $x \in A$.

Subcase 1.1: $x \neq a$. Then $x \in B \subseteq (B)_I$.

Subcase 1.2: $x = a$. By assumption, we have $x = a + bb + ShS \subseteq BB \cup SBS \subseteq (B)_I$.

**Case 2:** $x \in AA$. Then $x = a_i a_j$ for some $a_i, a_j \in A$.

Subcase 2.1: $a_i \neq a$ and $a_j \neq a$. Then $a_i a_j \in B$. We have $x = a_i a_j \in BB \subseteq (B)_I$.

Subcase 2.2: $a_i = a$ and $a_j \neq a$. By assumption and $a_j \in B$, we have $x = a_i a_j = a a \in (bb \cup ShS)B \subseteq (BB \cup SBS)B = BBB \cup SBSB \subseteq SBS \cup SBS = SBS \subseteq (B)_I$.

Subcase 2.3: $a_i \neq a$ and $a_j = a$. Then $a_j \in B$ and by assumption, we have $x = a_i a_j = a a \in B \subseteq BB \cup SBS \subseteq (BB \cup SBS) = BBB \cup BSBS \subseteq SBS \cup SBS = SBS \subseteq (B)_I$.

Subcase 2.4: $a_i = a$ and $a_j = a$. By assumption, we have $x = a_i a_j = a a \in (bb \cup ShS)B = bbbb \cup ShSb \cup ShSb \cup ShSb \subseteq BBB \cup BSBS \cup SBSB \cup SBS$.

**Case 3:** $x \in SAS$. Then $x = s_i s_j$ for some $s_i, s_j \in S$ and $a_i \in A$.  


Subcase 3.1: \(a_i \neq a\). Then \(a_i \in B\). We have \(x = s_i a_i s_j \in SBS \subseteq (B)_i\).

Subcase 3.2: \(a_i = a\). By assumption, we have

\[x = s_i a_i s_j \in S(bb \cup SbS)S \subseteq S(BB \cup SBS)S = (SBB \cup SSBS)S = SBS \cup SSBS \subseteq (B)_i\].

So, we obtain \(S \subseteq (B)_i\). This implies \((B)_i = S\), which is a contradiction since \(A\) is an interior base of \(S\). Thus \(a = b\).

**Lemma 2.5.** Let \(A\) be an interior base of a semigroup \(S\), and let \(a, b, c \in A\). If \(a \in cb \cup SbS\), then \(a = b\) or \(a = c\).

**Proof.** Assume that \(a \in cb \cup SbS\). Suppose that \(a \neq b\) and \(a \neq c\). Setting \(B = A \setminus \{a\}\). Then \(B \subseteq A\). Since \(a \neq b\) and \(a \neq c\), we have \(b, c \in B\). To show that \((A)_i \subseteq (B)_i\). Let \(x \in (A)_i\). Then \(x \in A \cup AA \cup SAS\). There are three cases to consider:

Case 1: \(x \in A\).

Subcase 1.1: \(x \neq a\). Then \(x \in B \subseteq (B)_i\).

Subcase 1.2: \(x = a\). By assumption, we have \(x = a \in cb \cup SbS \subseteq BB \cup SBS \subseteq BB \cup SBS \subseteq (B)_i\).

Case 2: \(x \in AA\). Then \(x = a_i a_j\) for some \(a_i, a_j \in A\).

Subcase 2.1: \(a_i \neq a\) and \(a_j \neq a\). Then \(a_i, a_j \in B\). We have \(x = a_i a_j \in BB \subseteq (B)_i\).

Subcase 2.2: \(a_i = a\) and \(a_j \neq a\). By assumption and \(a_j \neq a\), we have

\[x = a_i a_j \in (cb \cup SbS)B \subseteq (BB \cup SBS)B = BBB \cup SBSBB \subseteq SBS \subseteq (B)_i\].

Subcase 2.3: \(a_i \neq a\) and \(a_j = a\). Then \(a_i \in B\) and by assumption, we have

\[x = a_i a_j \in (cb \cup SbS)B \subseteq (B) \subseteq (B)_i\].

Subcase 2.4: \(a_i = a\) and \(a_j = a\). By assumption, we have

\[x = a_i a_j \in (cb \cup SbS)(cb \cup SbS) = cbcb \cup cbSbS \subseteq SBS \subseteq (B)_i\].

Case 3: \(x \in SAS\). Then \(x = s_i a_i s_j\) for some \(s_i, s_j \in S\) and \(a_i \in A\).

Subcase 3.1: \(a_i \neq a\). Then \(a_i \in B\). We have \(x = s_i a_i s_j \in SBS \subseteq (B)_i\).

Subcase 3.2: \(a_i = a\). By assumption, we have

\[x = s_i a_i s_j \in S(cb \cup SbS)S \subseteq S(BB \cup SBS)S = SBS \cup SSBS \subseteq SBS \subseteq (B)_i\].

From both cases, we obtain \((A)_i \subseteq (B)_i\). Since \(A\) is an interior base of \(S\), we have \(S = (A)_i \subseteq (B)_i \subseteq S\). Thus \(S = (B)_i\), which is a contradiction. Therefore, \(a = b\) or \(a = c\).

To give a characterization when a non-empty subset of a semigroup is an interior base of a semigroup, we need the concept of a quasi-order defined as follows:

**Definition 2.6.** Let \(S\) be a semigroup. Define a quasi-order \(\leq_i\) on \(S\) by, for any \(a, b \in S\),

\[a \leq_i b \iff (a)_i \subseteq (b)_i\].

The following example shows that the order \(\leq_i\) defined above is not, in general, a partial order.

**Example 2.7.** From Example 2.2, we have that \((b)_i \subseteq (c)_i\) (i.e., \(b \leq_i c\)) and \((c)_i \subseteq (b)_i\) (i.e., \(c \leq_i b\)), but \(b \neq c\). Thus \(\leq_i\) is not a partial order on \(S\).

**Lemma 2.8.** Let \(A\) be an interior base of a semigroup \(S\). If \(a, b \in A\) such that \(a \neq b\), then neither \(a \leq_i b\) nor \(b \leq_i a\).

**Proof.** Assume that \(a, b \in A\) such that \(a \neq b\). Suppose that \(a \leq_i b\). Then \(a \in (a)_i \subseteq (b)_i\). Since \(a \in (b)_i = b \cup bb \cup SbS\) and \(a \neq b\), so we have \(a \in bb \cup SbS\). By Lemma 2.4, \(a = b\). This is a contradiction. The case \(b \leq_i a\) can be proved similarly. Thus \(a \leq_i b\) and \(b \leq_i a\) are false.

**Lemma 2.9.** Let \(A\) be an interior base of a semigroup \(S\). Let \(a, b, c \in A\) and \(s \in S\).
(1) If \( a \in bc \cup bcbc \cup SbcS \), then \( a = b \) or \( a = c \).
(2) If \( a \in sbcs \cup sbcssbcs \cup SbscsS \), then \( a = b \) or \( a = c \).

**Proof.** (1) Assume that \( a \in bc \cup bcbc \cup SbcS \) and suppose that \( a \neq b \) and \( a \neq c \). Let \( B = A \setminus \{a\} \). Then \( B \subset A \). Since \( a \neq b \) and \( a \neq c \), we have \( b, c \in B \). We will show that \( (A)_I \subseteq (B)_I \). It suffices to show that \( A \subseteq (B)_I \). Let \( x \in A \). If \( x \neq a \), then \( x \in B \subseteq (B)_I \). So \( x \in (B)_I \). If \( x = a \), then by assumption, we have
\[
x = a \in bc \cup bcbc \cup SbcS \subseteq BB \cup BBBS \subseteq BB \cup SBS \subseteq (B)_I.
\]
Thus \( A \subseteq (B)_I \). This implies \( (A)_I \subseteq (B)_I \). So \( S = (A)_I \subseteq (B)_I \subseteq S \). Hence, \( S = (B)_I \). This is a contradiction. Therefore, \( a = b \) or \( a = c \).

(2) Assume that \( a \in sbcs \cup sbcssbcs \cup SbscsS \), and suppose that \( a \neq b \) and \( a \neq c \). Let \( B = A \setminus \{a\} \). Then \( B \subset A \). Since \( a \neq b \) and \( a \neq c \), we have \( b, c \in B \). We will show that \( (A)_I \subseteq (B)_I \). It suffices to show that \( A \subseteq (B)_I \). Let \( x \in A \). If \( x \neq a \), then \( x \in B \subseteq (B)_I \). So \( x \in (B)_I \). If \( x = a \), then by assumption, we have
\[
x = a \in sbcs \cup sbcssbcs \cup SbscsS \subseteq SBBS \cup SBBSSBS \cup SBBSS \subseteq SBS \subseteq (B)_I.
\]
Thus \( x \in (B)_I \), and so \( A \subseteq (B)_I \). This implies \( (A)_I \subseteq (B)_I \). So \( S = (A)_I \subseteq (B)_I \subseteq S \), and hence, \( S = (B)_I \). This is a contradiction. Therefore, \( a = b \) or \( a = c \).

**Lemma 2.10.** Let \( A \) be an interior base of a semigroup \( S \).

(1) For any \( a, b, c \in A \), if \( a \neq b \) and \( a \neq c \), then \( a \nleq bc \).
(2) For any \( a, b, c \in A \) and \( s \in S \), if \( a \neq b \) and \( a \neq c \), then \( a \nleq sbcs \).

**Proof.** (1) Let \( a, b, c \in A \), where \( a \neq b \) and \( a \neq c \). Suppose that \( a \leq bc \). Then \( a \in (A)_I \subseteq (bc)_I \). Since \( a \in (bc)_I \), by Lemma 2.9(1), we have \( a = b \) or \( a = c \). This contradicts to assumption. Thus \( a \nleq bc \).

(2) Let \( a, b, c \in A \) and \( s \in S \), where \( a \neq b \) and \( a \neq c \). Suppose that \( a \leq sbcs \). Then \( a \in (A)_I \subseteq (sbcs)_I = sbcs \cup sbcssbcs \cup SbscsS \). By Lemma 2.9(2), it follows that \( a = b \) or \( a = c \). This contradicts to assumption. Thus \( a \nleq sbcs \).

**Lemma 2.11.** Let \( A \) be an interior base of a semigroup \( S \). For any \( a, b \in A \) and \( s_1, s_2 \in S \), if \( a \neq b \), then \( a \nleq s_1bs_2 \).

**Proof.** Let \( a, b \in A \) and \( s_1, s_2 \in S \). Assume that \( a \neq b \) and suppose that \( a \leq s_1bs_2 \). Then \( a \in (A)_I \subseteq (s_1bs_2)_I \), and so \( a \in (s_1bs_2)_I = s_1bs_2 \cup s_1s_2s_1s_2 \cup Ss_1bs_2 \). Setting \( B = A \setminus \{a\} \). Then \( B \subset A \). Since \( a \neq b \), we have \( b, c \in B \). We will show that \( (A)_I \subseteq (B)_I \). It suffices to show that \( A \subseteq (B)_I \). Let \( x \in A \). If \( x \neq a \), then \( x \in B \subseteq (B)_I \). So \( x \in (B)_I \). If \( x = a \), then by assumption, we have
\[
x = a \in s_1bs_2 \cup s_1s_2s_1s_2 \cup Ss_1bs_2 \subseteq SBS \cup SBBSSBS \cup SBBSS \subseteq SBS \subseteq (B)_I.
\]
Thus \( x \in (B)_I \), and so \( A \subseteq (B)_I \). This implies \( (A)_I \subseteq (B)_I \). So \( S = (A)_I \subseteq (B)_I \subseteq S \), and hence, \( S = (B)_I \). This is a contradiction. Therefore, \( a \nleq s_1bs_2 \).

The following theorem characterizes when a non-empty subset of a semigroup \( S \) is an interior base of \( S \).

**Theorem 2.12.** A non-empty subset \( A \) of a semigroup \( S \) is an interior base of \( S \) if and only if \( A \) satisfies the following conditions:

(1) For any \( x \in S \),
   (1.1) there exists \( a \in A \) such that \( x \leq a \); or
   (1.2) there exist \( a_1, a_2 \in A \) such that \( x \leq a_1a_2 \); or
   (1.3) there exist \( a_1 \in A \), \( s_1, s_2 \in S \) such that \( x \leq s_1a_1s_2 \).
(2) For any \( a, b, c \in A \), if \( a \neq b \) and \( a \neq c \), then \( a \nleq bc \).
(3) For any \( a, b \in A \) and \( s_1, s_2 \in S \), if \( a \neq b \), then \( a \nleq s_1bs_2 \).
Proof. Assume that $A$ is an interior base of $S$. Then $S=(A)_I$. To show that (1) holds, let $x \in S$. We have $x \in (A)_I = A \cup AA \cup SAS$. There are three cases to consider:

Case 1: $x \in A$. Then $x \leq_I x$.

Case 2: $x \in AA$. Then $x = a_1 a_2$ for some $a_1, a_2 \in A$. This implies $(x)_I = (a_1 a_2)_I$. Hence, $x \leq_I a_1 a_2$.

Case 3: $x \in SAS$. Then $x = s_1 a_1 s_2$ for some $a_1, s_1, s_2 \in S$. We obtain $(x)_I = (s_1 a_1 s_2)_I$. Hence, $x \leq_I s_1 a_1 s_2$.

From both cases, we conclude that the condition (1) holds. The validity of (2) and (3) follow, respectively, from Lemma 2.10(1) and Lemma 2.11.

Conversely, assume that the conditions (1), (2) and (3) hold. We will show that $A$ is an interior base of $S$. First, we claim that $S = (A)_I$. Clearly, $A \subseteq S$. By (1.1), we have $S \subseteq A$. From (1.2), we obtain $S \subseteq AA$. By (1.3), we have $S \subseteq SAS$. So $S \subseteq A \cup AA \cup SAS = (A)_I$. Thus $S = (A)_I$. Next, to show that $A$ is a minimal subset of $S$ with the property $S = (A)_I$, we suppose that $S = (B)_I$ for some $B \subset A$. Since $B \subset A$, there exists $x \in A$ such that $x \not\in B$. Since $x \in A \subseteq S = (B)_I$ and $x \not\in B$, we have $x \in BB \cup SBS$. There are two cases to consider:

Case 1: $x \in BB$. Then $x = a_1 a_2$ for some $a_1, a_2 \in B$. We have $a_1, a_2 \in A$. Since $x \not\in B$, so $x \neq a_1$ and $x \neq a_2$.

Since $x = a_1 a_2$, we obtain $(x)_I \subseteq (a_1 a_2)_I$. Thus $x \leq_I a_1 a_2$. This contradicts to (2).

Case 2: $x \in SBS$. Then $x = s_1 a_1 s_2$ for some $s_1, s_2 \in S$ and $a_1 \in B$. We have $a_1 \in A$. Since $x \not\in B$, so $x \neq a_1$.

Since $x = s_1 a_1 s_2$, we obtain $(x)_I \subseteq (s_1 a_1 s_2)_I$. Thus $x \leq_I s_1 a_1 s_2$. This contradicts to (3).

Therefore, $A$ is an interior base of $S$.

In Example 2.3, we have that $\{0, 1\}$ is an interior base of $S$ where as it is not a subsemigroup of $S$. The following theorem we find a condition for an interior base is a subsemigroup.

Theorem 2.13. Let $A$ be an interior base of a semigroup $S$. Then $A$ is a subsemigroup of $S$ if and only if for any $a, b \in A$, $ab = a$ or $ab = b$.

Proof. Assume that $A$ is a subsemigroup of $S$. Suppose that $ab \neq a$ and $ab \neq b$. Let $c = ab$. Then $c \neq a$ and $c \neq b$. Since $c = ab \in ab \cup SBS$, by Lemma 2.5, we have $c = a$ or $c = b$. This is a contradiction.

The converse statement is obvious.

### 3. Conclusions

In this paper, we introduce the concept of interior bases of a semigroup by using the concepts of bases and interior ideals of a semigroup. The main theorems of this paper are Theorem 2.12 and Theorem 2.13. In Theorem 2.12, we give the necessary and sufficient condition for a non-empty subset of a semigroup to be an interior base. In Theorem 2.13, we give the necessary and sufficient condition for an interior base of a semigroup to be a subsemigroup.

In the future work, we can introduce other bases of a semigroup by using concepts of bases and other ideals of a semigroup.

### References


