บนฐานคู่ของแกมมากึ่งไฮเพอร์กรุปอันดับ
On bi-bases of ordered Γ-semihypergroups

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บทคัดย่อ

จุดประสงค์หลักของบทความนี้ บนพื้นฐานของแนวคิดของไบแกมมาไฮเพอร์ไอดีลที่ก่อกำเนิดโดยเซตย่อยไม่ว่าง เราแนะนำแนวคิดของบนฐานคู่ของแกมมากึ่งไฮเพอร์กรุปอันดับ เราอธิบายลักษณะของเซตย่อยไม่ว่างของแกมมากึ่งไฮเพอร์กรุปอันดับผลการวิจัยได้จากการขยายแนวคิดบนแกมมากึ่งกรุป

คำสำคัญ: แกมมากึ่งไฮเพอร์กรุปอันดับ ไบแกมมาไฮเพอร์ไอดีล ไบเบส อันดับควอซี หรือ ควอซี-อันดับ หรือ ควอซี-ออเดอร์

Abstract

The primary purpose of this paper was based on the notion of bi--hyperideal generated by non-empty subsets of an ordered -semihypergroups. The notion of bi-bases on was introduced and the quasi-order was defined by the principal of bi--hyperideal. Moreover, a non-empty subset was a bi-base and it was characterized. The results were obtained by extending the concept on -semigroup.

Keywords: Ordered Γ-semihypergroups, Bi -Γ-hyperideals, Bi-bases, Quasi-order.

1. Introduction and Preliminaries

The algebraic hyperstructure notion was introduced in 1934 by the Freanch mathematician Marty. The notion of two-sided base of a semigroup was introduced by I. Fabrici [1]. The results [1] have extended to ordered semigroups by T. Changpas and P. Summaprab [8]. In 2017, T. Changpas and P. Kummoon studied the notion of bi-base of a semigroup [4] and bi-base of Γ-semigroup [5]. The main purpose of this paper is to introduce the concept to extend the results to on bi-bases of ordered Γ-semihypergroups.

Let H be a non-empty set. A mapping $\circ : H \times H \to P^*(H)$ where $P^*(H)$ denotes the family of all non-empty subsets of H. If A and B are two non-empty subsets of H and $h \in H$, then we denote $A \circ B = \bigcup_{a \in A, b \in B} a \circ b, x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$.

Definition 1.1 [3] Let H and Γ be two non-empty sets. H is called a Γ**-semihypergroup**. If every $\gamma \in \Gamma$ is a hyperoperation on H, $x\gamma y \subseteq H$ for every $x, y \in H$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in H$. We have $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Let A and B be two non-empty subsets of H and $\gamma \in \Gamma$. We define

$$A\gamma B = \bigcup_{a \in A, b \in B} a\gamma b \text{ and } A\Gamma B = \bigcup_{\gamma \in \Gamma} A\gamma B.$$

Definition 1.2 [3] An algebraic hyperstructure $(H, \Gamma, \leq)$ is called an **ordered** Γ**-semihypergroup**. If (H, Γ) is a Γ-semihypergroup and $(H, \leq)$ is a partially ordered set such that for any $x, y, z \in H$, $x \leq y$ implies $z\gamma x \leq z\gamma y$ and $x\gamma z \leq y\gamma z$. Here, $A \leq B$ means that for any $a \in A$, there exists $b \in B$ such that $a \leq b$, for all non-empty subsets A and B of H.

In what follows we denote and ordered Γ-semihypergroup $(H, \Gamma, \leq)$ by H unless otherwise specified.

The purpose of this paper is to introduce the concept of bi-bases of an ordered Γ-semihypergroup and extend some of bi-bases of Γ-semigroups results.

Definition 1.3 [3] A non-empty subset A of an ordered Γ-semihypergroup H is called a **sub Γ-semihypergroup** of H if $A\Gamma A \subseteq A$.

Notation 1.4 [2],[6] Let K be a non-empty subset of an ordered Γ-semihypergroup H. We define $(K] := \{x \in H \mid x \leq k$ for some $k \in K\}$. For $K = \{k\}$, we write $(k]$ instead of $(\{k\}]$.

If A and B are non-empty subsets of H, then we have

(1) $A \subseteq (A]$;

(2) $((A]] = (A]$;

(3) If $A \subseteq B$, then $(A] \subseteq (B]$;

(4) $(A]\Gamma(B] \subseteq (A\Gamma B]$;

(5) $((A]\Gamma(B]] = (A\Gamma B]$; and

(6) $(A] \cup (B] = (A \cup B]$.

Definition 1.5 [3] A sub Γ-semihypergroup B of an ordered Γ-semihypergroup H is called a **bi-Γ-hyperideal** of H if

(i) $B\Gamma H\Gamma B \subseteq B$ and

(ii) if $a \in B$, $b \leq a$ for $b \in H$ implies $b \in B$.

Lemma 1.6 *Let H be an ordered Γ-semihypergroup and let B_i be a sub Γ-semihypergroup of H. For all $i \in I$, If $\bigcap_{i\in I} B_i \neq \varnothing$, then $\bigcap_{i\in I} B_i$ is a sub Γ-semihypergroup of H.*

Proof. Assume that $\bigcap_{i\in I} B_i \neq \varnothing$. Let $a,b \in \bigcap_{i\in I} B_i$ for all $i \in I$. Since B_i is a sub Γ-semihypergroup for all $i \in I$, we obtain $a\gamma b \in B_i$ for all $i \in I$ and $\gamma \in \Gamma$. Thus $a\gamma b \in \bigcap_{i\in I} B_i \neq \varnothing$. Hence $\bigcap_{i\in I} B_i$ is a sub Γ-semihypergroup of H.

Proposition 1.7 *Let H be an ordered Γ-semihypergroup and B_i be a bi-Γ-hyperideal of H. For each i in an indexed set I, if $\bigcap_{i\in I} B_i \neq \varnothing$, then $\bigcap_{i\in I} B_i$ is a bi-Γ-hyperideal of H.*

Proof. Assume that $\bigcap_{i\in I} B_i \neq \varnothing$. By Lemma 1.6, we have $\bigcap_{i\in I} B_i$ is a sub Γ-semihypergroup of H. Suppose that $\bigcap_{i\in I} B_i \neq \varnothing$. Let $a \in (\bigcap_{i\in I} B_i)\Gamma H\Gamma(\bigcap_{i\in I} B_i)$. We have $a \in b_1\gamma_1 h\gamma_2 b_2$ for some $b_1, b_2 \in \bigcap_{i\in I} B_i$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$. Since $b_1, b_2 \in \bigcap_{i\in I} B_i$, so we obtain $b_1, b_2 \in B_i$ for all $i \in I$. For any $i \in I$, we have B_i is a bi-Γ-hyperideal of H.

Hence $a \in b_1\gamma_1 h\gamma_2 b_2 \subseteq B_i$ for all $i \in I$.

Thus $(\bigcap_{i\in I} B_i)\Gamma H\Gamma(\bigcap_{i\in I} B_i) \subseteq \bigcap_{i\in I} B_i$. Next we show that, if $a \in \bigcap_{i\in I} B_i$ and $c \in H$ such that $c \leq a$, then $c \in \bigcap_{i\in I} B_i$. Let $a \in \bigcap_{i\in I} B_i$ and $c \in H$ such that $c \leq a$. Since $a \in \bigcap_{i\in I} B_i$ and B_i is a bi-Γ-hyperideal of H for all $i \in I$, we have $c \in B_i$ for all $i \in I$. Thus $c \in \bigcap_{i\in I} B_i$ for all $i \in I$. Therefore $\bigcap_{i\in I} B_i$ is a bi-Γ-hyperideal of H.

Notation 1.8 Let A be a non-empty subset of an ordered Γ-semihypergroup H. Then intersection of all bi-Γ-hyperideals of H containing A is the smallest bi-Γ-hyperideal of H generated by A and denoted by $B_H(A)$. In particular, for $A = \{a\}$, we write $B_H(\{a\})$ by $B_H(a)$.

Proposition 1.9 *Let A be a non-empty subset of an ordered Γ-semihypergroup H. Then $B_H(A) = (A \cup A\Gamma A \cup A\Gamma H\Gamma A]$.*

Proof. Let $B = (A \cup A\Gamma A \cup A\Gamma HA]$. It is clear that

$$\begin{aligned} B\Gamma B &= (A \cup A\Gamma A \cup A\Gamma H\Gamma A]\Gamma(A \cup A\Gamma A \cup A\Gamma H\Gamma A] \\ &\subseteq \left(A \cup A\Gamma A \cup A\Gamma H\Gamma A)\Gamma(A \cup A\Gamma A \cup A\Gamma H\Gamma A)\right] \\ &\subseteq (A\Gamma A \cup A\Gamma H\Gamma A] \subseteq B. \end{aligned}$$

Hence B is a sub Γ-semihypergroup of H. Next, we see that

$$\begin{aligned} B\Gamma H\Gamma B &= (A \cup A\Gamma A \cup A\Gamma H\Gamma A]\Gamma H\Gamma(A \cup A\Gamma A \cup A\Gamma H\Gamma A] \\ &= (A \cup A\Gamma A \cup A\Gamma H\Gamma A]\Gamma(H]\Gamma(A \cup A\Gamma A \cup A\Gamma H\Gamma A] \\ &\subseteq \left(A \cup A\Gamma A \cup A\Gamma H\Gamma A)\Gamma H\Gamma(A \cup A\Gamma A \cup A\Gamma H\Gamma A)\right] \\ &\subseteq (A\Gamma H\Gamma A] \end{aligned}$$

$\subseteq B,$ we infer that $B\Gamma H\Gamma B \subseteq B.$

Clearly, If $a \in (A \cup A\Gamma A \cup A\Gamma H\Gamma A]$ and $x \in H$ such that $x \leq a,$ then $x \in ((A \cup A\Gamma A \cup A\Gamma H\Gamma A]]$ $= (A \cup A\Gamma A \cup A\Gamma H\Gamma A].$ Therefore, B is a bi-Γ-hyperideal of H containing $A.$ Suppose that there exists C such that C is a bi-Γ-hyperideal of H containing A and $C \subseteq B.$ Since C is a bi-Γ-hyperideal of H containing $A,$ we have $A \subseteq C,$ $A\Gamma H\Gamma A \subseteq C\Gamma H\Gamma C \subseteq C$ and $A\Gamma A \subseteq C\Gamma C \subseteq C.$ Hence, $B = (A \cup A\Gamma A \cup A\Gamma H\Gamma A] \subseteq C.$ This means that $B = C.$ Hence B is the smallest bi-Γ-hyperideal of H containing $A.$ Therefore, $B_H(A) = (A \cup A\Gamma A \cup A\Gamma H\Gamma A].$

2. Main Results

In this section, we study properties of bi-base in an ordered Γ-semihypergroups.

Definition 2.1 Let H be an ordered Γ-semihypergroup. A subset B of H is called a **bi- base** of H if it satisfies the two following conditions:

(1) $H = B_H(B)$ (i.e. $H = (B \cup B\Gamma B \cup B\Gamma H\Gamma B]$);

(2) if A a subset of B such that $H = B_H(A)$ then $A = B.$

Example 2.2 Let $H = \{a,b,c,d\}$ and $\Gamma = \{\gamma,\beta\}$ be the sets of binary hyperoperations defined as follows:

γ	a	b	c	d
a	a	$\{a,b\}$	$\{c,d\}$	d
b	$\{a,b\}$	b	$\{c,d\}$	d
c	$\{c,d\}$	$\{c,d\}$	c	d
d	d	d	d	d

β	a	b	c	d
a	a	$\{a,b\}$	$\{c,d\}$	d
b	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	d
c	$\{c,d\}$	$\{c,d\}$	c	d
d	d	d	d	d

and

$\leq := \{(a,a),(a,b),(b,b),(c,b),(c,c),(c,d),(d,b),(d,d)\}.$

In [7] H is an ordered Γ-semihypergroups. Consider $B_1 = \{a\}$ and $B_2 = \{b\},$ we have B_1 and B_2 are bi-bases of H But $B_1' = \{a,b\}$ is not a bi-base of $H.$

Lemma 2.3 *Let* B *be a bi-base of an ordered* Γ*-semihypergroup* H *and* $a,b \in B.$
If $a \in (b\Gamma b \cup b\Gamma H\Gamma b],$ *then* $a = b.$

Proof. Assume that $a \in (b\Gamma b \cup b\Gamma H\Gamma b]$ and suppose that $a \neq b.$ Let $A = B \setminus \{a\}.$ It is clearly seen that $A \subset B.$ Since $a \neq b,$ so $b \in A.$ We will show that $B_H(A) = H.$ Clearly, $B_H(A) \subseteq H.$ Next, we show that $H \subseteq B_H(A).$ Let $x \in H.$ By hypothesis, we have $B_H(B) = H$ and so $x \in (B \cup B\Gamma B \cup B\Gamma H\Gamma B].$ Since $x \in (B \cup B\Gamma B \cup B\Gamma H\Gamma B],$ we have $x \leq y$ for some $y \in B \cup B\Gamma B \cup B\Gamma H\Gamma B.$ We can consider the three following cases.

Case 1: $y \in B.$ There are two subcases to consider.

Subcase 1.1: $y \neq a.$ Thus

$$y \in B \setminus \{a\} = A \subseteq B_H(A).$$

Subcase 1.2: $y = a.$ By assumption, we have

$$y = a \in (b\Gamma b \cup b\Gamma H\Gamma b] \subseteq (A\Gamma A \cup A\Gamma H\Gamma A] \subseteq B_H(A).$$

Case2: $y \in B\Gamma B.$ Hence $y \in b_1\gamma b_2$ for some $b_1, b_2 \in B.$ There are four subcases to consider.

Subcase 2.1: $b_1 = a$ and $b_2 = a.$ By assumption, so we have the following:

$$\begin{aligned} y \in b_1\gamma b_2 = a\gamma a &\subseteq (b\Gamma b \cup b\Gamma H\Gamma b]\Gamma(b\Gamma b \cup b\Gamma H\Gamma b] \\ &\subseteq \big((b\Gamma b \cup b\Gamma H\Gamma b)\Gamma(b\Gamma b \cup b\Gamma H\Gamma b)\big] \\ &= (b\Gamma b\Gamma b\Gamma b \cup b\Gamma b\Gamma b\Gamma H\Gamma b \cup b\Gamma H\Gamma b\Gamma b\Gamma b \cup b\Gamma H\Gamma b\Gamma b\Gamma H\Gamma b] \\ &\subseteq (A\Gamma A\Gamma A\Gamma A \cup A\Gamma A\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma A\Gamma H\Gamma A] \\ &\subseteq (A\Gamma H\Gamma A] \\ &\subseteq B_H(A). \end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a.$ By assumption and $A = B \setminus \{a\},$ we have

$$\begin{aligned} y \in b_1\gamma b_2 = b_1\gamma a &\subseteq (B \setminus \{a\})\Gamma(b\Gamma b \cup b\Gamma H\Gamma b] \\ &\subseteq (B \setminus \{a\}]\Gamma(b\Gamma b \cup b\Gamma H\Gamma b] \\ &\subseteq \big((B \setminus \{a\})\Gamma(b\Gamma b \cup b\Gamma H\Gamma b)\big] \\ &= \big((B \setminus \{a\})\Gamma b\Gamma b \cup (B \setminus \{a\})\Gamma b\Gamma H\Gamma b\big] \\ &\subseteq (A\Gamma A\Gamma A \cup A\Gamma A\Gamma H\Gamma A] \end{aligned}$$

$\subseteq (A\Gamma H\Gamma A]$
$\subseteq B_H(A).$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a.$ By assumption and $A = B \setminus \{a\},$ we have

$$y \in b_1\gamma b_2 = a\gamma b_2 \subseteq (b\Gamma b \cup b\Gamma H\Gamma b]\Gamma(B \setminus \{a\}) \subseteq (b\Gamma b \cup b\Gamma H\Gamma b]\Gamma(B \setminus \{a\}] \subseteq \left.(b\Gamma b \cup b\Gamma H\Gamma b)\Gamma(B \setminus \{a\})\right] = \left. b\Gamma b\Gamma(B \setminus \{a\}) \cup b\Gamma H\Gamma b\Gamma(B \setminus \{a\})\right] \subseteq (A\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma A] \subseteq (A\Gamma H\Gamma A] \subseteq B_H(A).$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a.$ By assumption and $A = B \setminus \{a\}.$ Then $y \in b_1\gamma b_2$
$\subseteq (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq B_H(A).$ 5

Case 3: $y \in B\Gamma H\Gamma B.$ Hence $y \in b_3\gamma_1 h\gamma_2 b_4$ for some $b_3, b_4 \in B,$ $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H.$ There are four subcases to consider.

Subcase 3.1: $b_3 = a$ and $b_4 = a.$ By assumption, we have

$$y \in b_3\gamma_1 h\gamma_2 b_4 = a\gamma_1 h\gamma_2 a \subseteq (b\Gamma b \cup b\Gamma H\Gamma b]\Gamma H\Gamma(b\Gamma b \cup b\Gamma H\Gamma b] = (b\Gamma b \cup b\Gamma H\Gamma b]\Gamma(H]\Gamma(b\Gamma b \cup b\Gamma H\Gamma b] \subseteq \left.(b\Gamma b \cup b\Gamma H\Gamma b)\Gamma H\Gamma(b\Gamma b \cup b\Gamma H\Gamma b)\right] = (b\Gamma b\Gamma H\Gamma b\Gamma b \cup b\Gamma b\Gamma H\Gamma b\Gamma H\Gamma b \cup b\Gamma H\Gamma b\Gamma H\Gamma b\Gamma b \cup b\Gamma H\Gamma b\Gamma H\Gamma b\Gamma H\Gamma b] \subseteq (A\Gamma A\Gamma H\Gamma A\Gamma A \cup A\Gamma A\Gamma H\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A\Gamma H\Gamma A] \subseteq (A\Gamma H\Gamma A] \subseteq B_H(A).$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a.$ By assumption and $A = B \setminus \{a\},$ we have

$$y \in b_3\gamma_1 h\gamma_2 b_4 = b_3\gamma_1 h\gamma_2 a \subseteq (B \setminus \{a\})\Gamma H\Gamma(b\Gamma b \cup b\Gamma H\Gamma b] \subseteq (B \setminus \{a\}]\Gamma(H]\Gamma(b\Gamma b \cup b\Gamma H\Gamma b] \subseteq \left.(B \setminus \{a\})\Gamma H\Gamma(b\Gamma b \cup b\Gamma H\Gamma b)\right] = \left.(B \setminus \{a\})\Gamma H\Gamma b\Gamma b \cup (B \setminus \{a\})\Gamma H\Gamma b\Gamma H\Gamma b\right] \subseteq (A\Gamma H\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A] \subseteq (A\Gamma H\Gamma A] \subseteq B_H(A).$$

Subcase 3.3: $b_3 = a$ and $b_4 \neq a.$ By assumption and $A = B \setminus \{a\},$ we have

$$y \in b_3\gamma_1 h\gamma_2 b_4 = a\gamma_1 h\gamma_2 b_4 \subseteq (b\Gamma b \cup b\Gamma H\Gamma b]\Gamma H\Gamma(B \setminus \{a\}) \subseteq (b\Gamma b \cup b\Gamma H\Gamma b]\Gamma(H]\Gamma(B \setminus \{a\}] \subseteq \left.(b\Gamma b \cup b\Gamma H\Gamma b)\Gamma H\Gamma(B \setminus \{a\})\right] = \left.(b\Gamma b\Gamma H\Gamma(B \setminus \{a\}) \cup b\Gamma H\Gamma b\Gamma H\Gamma(B \setminus \{a\})\right] \subseteq (A\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A] \subseteq (A\Gamma H\Gamma A] \subseteq B_H(A).$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a.$ By assumption and $A = B \setminus \{a\},$ we have

$$y \in b_3\gamma_1 h\gamma_2 b_4 \subseteq (B \setminus \{a\})\Gamma H\Gamma(B \setminus \{a\}) = A\Gamma H\Gamma A \subseteq B_H(A).$$

By case 1,2 and 3 we have $H \subseteq B_H(A).$ This implies that $B_H(A) = H.$ This is a contradiction. Therefore $a = b.$

Lemma 2.4 *Let B be a bi-base of an ordered Γ-semihypergroup H and $a, b, c \in B.$ If $a \in (c\Gamma b \cup c\Gamma H\Gamma b]$ then $a = b$ or $a = c.$*

Proof. Assume that $a \in (c\Gamma b \cup c\Gamma H\Gamma b].$ Suppose that $a \neq b,$ $a \neq c.$ Let $A = B \setminus \{a\}.$ Since $a \neq b$ and $a \neq c,$ we have $b, c \in A.$

We will show that $B_H(A) = H.$ Clearly, $B_H(A) \subseteq H.$ Let $x \in H,$ we need to prove only that $H \subseteq B_H(A).$ Since B is a bi-base of $H,$ we have $x \in (B \cup B\Gamma B \cup B\Gamma H\Gamma B].$

Since $x \in (B \cup B\Gamma B \cup B\Gamma H\Gamma B],$ then $x \leq y$ for some $y \in B \cup B\Gamma B \cup B\Gamma H\Gamma B.$ We can consider the three following cases.

Case 1: $y \in B.$ There are two subcases to consider.

Subcase 1.1: $y \neq a.$ Then

$$y \in B \setminus \{a\} = A \subseteq B_H(A).$$

Subcase 1.2: $y = a.$ By assumption, we have

$$y = a \in (c\Gamma b \cup c\Gamma H\Gamma b] \subseteq (A\Gamma A \cup A\Gamma H\Gamma A] \subseteq B_H(A).$$

Case 2: $y \in B\Gamma B$. Then $y \in b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$. There are four subcases to consider.

Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption, we have

$$\begin{aligned} y \in b_1\gamma b_2 &= a\gamma a \\ &\subseteq (c\Gamma b \cup c\Gamma H\Gamma b]\Gamma(c\Gamma b \cup c\Gamma H\Gamma b] \\ &\subseteq ((c\Gamma b \cup c\Gamma H\Gamma b)\Gamma(c\Gamma b \cup c\Gamma H\Gamma b)] \\ &\subseteq (A\Gamma H\Gamma A] \\ &\subseteq B_H(A). \end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} y \in b_1\gamma b_2 = b_1\gamma a &\subseteq (B \setminus \{a\})\Gamma(c\Gamma b \cup c\Gamma H\Gamma b] \\ &\subseteq (B \setminus \{a\}]\Gamma(c\Gamma b \cup c\Gamma H\Gamma b] \\ &\subseteq ((B \setminus \{a\})\Gamma(c\Gamma b \cup c\Gamma H\Gamma b)] \\ &\subseteq (A\Gamma H\Gamma A] \\ &\subseteq B_H(A). \end{aligned}$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} y \in b_1\gamma b_2 &= a\gamma b_2 \\ &\subseteq (c\Gamma b \cup c\Gamma H\Gamma b]\Gamma(B \setminus \{a\}) \\ &\subseteq (c\Gamma b \cup c\Gamma H\Gamma b]\Gamma(B \setminus \{a\}] \\ &\subseteq ((c\Gamma b \cup c\Gamma H\Gamma b)\Gamma(B \setminus \{a\})] \\ &\subseteq (A\Gamma H\Gamma A] \\ &\subseteq B_H(A). \end{aligned}$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. By assumption and $A = B \setminus \{a\}$, hence

$$y \in b_1\gamma b_2 \subseteq (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq B_H(A).$$

Case 3: $y \in B\Gamma H\Gamma B$. Hence $y \in b_3\gamma_1 h\gamma_2 b_4$ for some $b_3, b_4 \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$. There are four subcases to consider.

Subcase 3.1: $b_3 = a$ and $b_4 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} y \in b_3\gamma_1 h\gamma_2 b_4 &= a\gamma_1 h\gamma_2 a \\ &\subseteq (c\Gamma b \cup c\Gamma H\Gamma b]\Gamma H\Gamma(c\Gamma b \cup c\Gamma H\Gamma b] \\ &\subseteq ((c\Gamma b \cup c\Gamma H\Gamma b)\Gamma H\Gamma(c\Gamma b \cup c\Gamma H\Gamma b)] \\ &\subseteq (A\Gamma H\Gamma A] \\ &\subseteq B_H(A). \end{aligned}$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} y \in b_3\gamma_1 h\gamma_2 b_4 &= b_3\gamma_1 h\gamma_2 a \\ &\subseteq (B \setminus \{a\})\Gamma H\Gamma(c\Gamma b \cup c\Gamma H\Gamma b] \\ &\subseteq (B \setminus \{a\}]\Gamma(H]\Gamma(c\Gamma b \cup c\Gamma H\Gamma b] \\ &\subseteq ((B \setminus \{a\})\Gamma H\Gamma(c\Gamma b \cup c\Gamma H\Gamma b)] \\ &\subseteq (A\Gamma H\Gamma A] \\ &\subseteq B_H(A). \end{aligned}$$

Subcase 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} y \in b_3\gamma_1 h\gamma_2 b_4 &= a\gamma_1 h\gamma_2 b_4 \\ &\subseteq (c\Gamma b \cup c\Gamma H\Gamma b]\Gamma H\Gamma(B \setminus \{a\}) \\ &\subseteq (c\Gamma b \cup c\Gamma H\Gamma b]\Gamma(H]\Gamma(B \setminus \{a\}] \\ &\subseteq ((c\Gamma b \cup c\Gamma H\Gamma b)\Gamma H\Gamma(B \setminus \{a\})] \\ &\subseteq (A\Gamma H\Gamma A] \\ &\subseteq B_H(A). \end{aligned}$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. By assumption and $A = B \setminus \{a\}$, we have

$$\begin{aligned} y \in b_3\gamma_1 h\gamma_2 b_4 &\subseteq (B \setminus \{a\})\Gamma H\Gamma(B \setminus \{a\}) \\ &= A\Gamma H\Gamma A \\ &\subseteq B_H(A). \end{aligned}$$

By case 1,2 and 3 we have $H \subseteq B_H(A)$. This implies $B_H(A) = H$. This is a contradiction. Therefore $a = b$.

To characterize when a non-empty subset of an ordered Γ-semihypergroup is a bi-bases of the ordered Γ-semihypergroup we need the quasi-ordered defined as follows :

Notation 2.5 Let H be an ordered Γ-semihypergroup. For any $a, b \in H$ define a **quasi-order** on H by $a \leq_b b \Leftrightarrow B_H(a) \subseteq B_H(b)$.

The following examples show that the order $\leq_b$ Defined above is not, ingenerl a partial order.

Example 2.6 From Example 2.2, we have that $B_H(a) \subseteq B_H(b)$ (i.e, $a \leq_b b$) and $B_H(b) \subseteq B_H(a)$ (i.e, $b \leq_b a$) but $a \neq b$. Thus $\leq_b$ is not a partial order on H.

If A is a bi-base of H. Then $B_H(A) = H$. Let $x \in H$. Then $x \in B_H(A)$ and so $x \in B_H(a)$ for some $a \in A$. This implies $B_H(x) \subseteq B_H(a)$. Hence $x \leq_b a$. Then we conclude that:

Let B be a non-empty subset of an ordered Γ-semihypergroup H. If B is a bi-base of H, then for any $x \in H$ there exists $a \in B$ such that $x \leq_b a$.

Lemma 2.7 *Let B be a bi-base of an ordered Γ-semihypergroup H. If $a,b \in B$ such that $a \neq b$, then neither $a \leq_b b$, nor $b \leq_b a$.*

Proof. Let $a,b \in B$ such that $a \neq b$. Suppose $a \leq_b b$. We set $A = B \setminus \{a\}$. Then $b \in A$.

Let $x \in H$. There exists $c \in B$ such that $x \leq_b c$. There are two cases to consider. If $c \neq a$, then $c \in A$. Thus $B_H(x) \subseteq B_H(c) \subseteq B_H(A)$. Hence $H = B_H(A)$. This is a contradiction. If $c = a$, then $x \leq_b b$. Hence $x \in B_H(A)$. There fore $H = B_H(A)$. This is a contradiction. The case $b \leq_b a$ is proved similarly.

Lemma 2.8 *Let B be a bi-base of an ordered Γ-semihypergroup H. Let $a,b,c \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$:*

(1) If $a \in (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c]$, then $a = b$ or $a = c$.

(2) If $a \in (b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c]$, then $a = b$ or $a = c$.

Proof. (1) Assume that $a \in (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c]$ and suppose that $a \neq b$ and $a \neq c$. Let $A = B \setminus \{a\}$. Then $A \subset B$. Since $a \neq b$ and $a \neq c$, we have $b,c \in A$. We will show that $B_H(B) \subseteq B_H(A)$, it suffices to show that $B = B_H(A)$. Let $x \in B$. if $x \neq a$, then $x \in A$. Hence $x \in B_H(A)$. If $x = a$, then by assumption we have

$$\begin{aligned} x = a &\in (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c] \\ &\subseteq (A\Gamma A \cup A\Gamma A\Gamma A\Gamma A \cup A\Gamma A\Gamma H\Gamma A\Gamma A] \\ &\subseteq (A\Gamma A \cup A\Gamma H\Gamma A] \\ &\subseteq B_H(A). \end{aligned}$$

Thus $B \subseteq B_H(A)$. This implies that $B_H(B) \subseteq B_H(A)$. Since B is a bi-base of H and $H = B_H(B) \subseteq B_H(A) \subseteq H$, we have $H = B_H(A)$. This is a contradiction.

(2) Assume that $a \in (b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c]$, and suppose that $a \neq b$ and $a \neq c$. Let $A = B \setminus \{a\}$. Then $A \subset B$. Since $a \neq b$ and $a \neq c$, we have $b,c \in A$. We will show that $B_H(B) \subseteq B_H(A)$, if suffices to show that $B \subseteq B_H(A)$. Let $x \in B$. If $x \neq a$, then $x \in A$. Hence $x \in B_H(A)$. If $x = a$, then by assumption we have

$$\begin{aligned} x = a &\in (b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c] \\ &\subseteq (A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A\Gamma H\Gamma A] \\ &\subseteq (A\Gamma H\Gamma A] \\ &\subseteq B_H(A). \end{aligned}$$

Thus $B \subseteq B_H(A)$. This implies that $B_H(B) \subseteq B_H(A)$. Since B is a bi-base of H and $H = B_H(B) \subseteq B_H(A) \subseteq H$, we have $H = B_H(A)$. This is a contradiction.

Lemma 2.9 *Let B be a bi-base of an ordered Γ-semihypergroups H.*

(1) For any $a,b,c \in B$ and $\gamma_1 \in \Gamma$ and if $a \neq b$ and $a \neq c$ then $a \not\leq_b b\gamma_1 c$.

(2) For any $a,b,c \in B$, $\gamma_2, \gamma_3 \in \Gamma$ and $h \in H$ if $a \neq b$ and $a \neq c$ then $a \not\leq_b b\gamma_2 h\gamma_3 c$.

Proof. (1) For any $a,b,c \in B$, $\gamma_1 \in \Gamma$ let $a \neq b$ and $a \neq c$. Suppose that $a \leq_b b\gamma_1 c$. We have

$$\begin{aligned} a \in B_H(a) &\subseteq B_H(b\gamma_1 c) \\ &= (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c]. \end{aligned}$$

Hence $a \in (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c]$. By Lemma 2.8(1), it follows that $a = b$ or $a = c$. This contradicts to assumption.

(2) For any $a,b,c \in B$, $\gamma_2, \gamma_3 \in \Gamma$ and $h \in H$, let $a \neq b$ and $a \neq c$. Suppose that $a \leq_b b\gamma_2 h\gamma_3 c$. We have

$$\begin{aligned} a \in B_H(a) &\subseteq B_H(b\gamma_2 h\gamma_3 c) \\ &= (b\gamma_2 h\gamma_3 c \cup b\gamma_2 h\gamma_3 c\Gamma b\gamma_2 h\gamma_3 c \cup b\gamma_2 h\gamma_3 c\Gamma H\Gamma b\gamma_2 h\gamma_3 c]. \end{aligned}$$

Hence $a \in (b\gamma_2 h\gamma_3 c \cup b\gamma_2 h\gamma_3 c\Gamma b\gamma_2 h\gamma_3 c \cup b\gamma_2 h\gamma_3 c\Gamma H\Gamma b\gamma_2 h\gamma_3 c]$. By Lemma 2.8(2), it follows that $a = b$ or $a = c$. This contradicts to assumption.

The following theorems characterizes when a non-empty subset of an ordered Γ-semihypergroups H is a bi-base of H.

Theorem 2.10 *A non-empty subset B of an ordered Γ-semihypergroups H is a bi-base if and only if B satisfies the following conditions:*

(1) For any $x \in H$

(1.1) there exists $b \in B$ such that $x \leq_b b$; or

(1.2) there exists $b_1, b_2 \in B$ and $\gamma \in \Gamma$ such that $x \leq_b b_1\gamma b_2$; or

(1.3) there exists $b_3, b_4 \in B, h \in H$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $x \leq_b b_3\gamma_1 h\gamma_2 b_4$.

(2) For any $a,b,c \in B$ and $\gamma_1 \in \Gamma$ if $a \neq b$ and $a \neq c$ then $a \not\leq_b b\gamma_1 c$.

(3) For any $a,b,c \in B, \gamma_2, \gamma_3 \in \Gamma$ and $h \in H$ if $a \neq b$ and $a \neq c$ then $a \not\leq_b b\gamma_2 h\gamma_3 c$.

Proof. Assume that B is a bi-base of H. Then $H = B_H(B)$. To show that (1) hold.

Let $x \in H$. Then $x \in (B \cup B\Gamma B \cup B\Gamma H\Gamma B]$. Since $x \in (B \cup B\Gamma B \cup B\Gamma H\Gamma B]$, we have $x \leq y$, for some $y \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$. We can consider the three following cases.

Case 1: $y \in B$. Thus $y = b$ for some $b \in B$. This implies $B_H(y) \subseteq B_H(b)$. Hence $y \leq_b b$. Since $x \leq y$ for some $y \in B_H(b)$, we have $x \in B_H(b)$. We will show that $B_H(x) \subseteq B_H(b)$.

Consider

$$\begin{aligned} x \cup x\Gamma x \cup x\Gamma H\Gamma x &\subseteq B_H(b) \cup B_H(b)\Gamma B_H(b) \cup B_H(b)\Gamma H\Gamma B_H(b) \\ &= (b \cup b\Gamma b \cup b\Gamma H\Gamma b] \cup (b \cup b\Gamma b \cup b\Gamma H\Gamma b]\Gamma(b \cup b\Gamma b \cup b\Gamma H\Gamma b] \cup (b \cup b\Gamma b \cup b\Gamma H\Gamma b]\Gamma H\Gamma(b \cup b\Gamma b \cup b\Gamma H\Gamma b] \\ &\subseteq (b \cup b\Gamma b \cup b\Gamma H\Gamma b] \\ &= B_H(b). \end{aligned}$$

Then we have $B_H(x) = (x \cup x\Gamma x \cup x\Gamma H\Gamma x]$

$$\subseteq (B_H(b)] = B_H(b).$$

This implies that $B_H(x) \subseteq B_H(b)$. Hence $x \leq_b b$.

Case 2: $y \in B\Gamma B$. Then $y \in b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$. This implies that $B_H(y) \subseteq B_H(b_1\gamma b_2)$. Hence $y \leq_b b_1\gamma b_2$. Since $x \leq y$ for some $y \in B_H(b_1\gamma b_2)$, we have $x \in B_H(b_1\gamma b_2)$. We will show that $B_H(x) \subseteq B_H(b_1\gamma b_2)$. Consider

$$\begin{aligned} x \cup x\Gamma x \cup x\Gamma H\Gamma x &\subseteq B_H(b_1\gamma b_2) \cup B_H(b_1\gamma b_2)\Gamma B_H(b_1\gamma b_2) \cup B_H(b_1\gamma b_2)\Gamma H\Gamma B_H(b_1\gamma b_2) \\ &\subseteq (b_1\gamma b_2 \cup b_1\gamma b_2\Gamma b_1\gamma b_2 \cup b_1\gamma b_2\Gamma H\Gamma b_1\gamma b_2] \\ &= B_H(b_1\gamma b_2). \end{aligned}$$

Then we have $B_H(x) = (x \cup x\Gamma x \cup x\Gamma H\Gamma x]$

$$\subseteq (B_H(b_1\gamma b_2)] = B_H(b_1\gamma b_2).$$

This implies $B_H(x) \subseteq B_H(b_1\gamma b_2)$. Hence $x \leq_b b_1\gamma b_2$.

Case 3: $y \in B\Gamma H\Gamma B$. Then $y \in b_3\gamma_1 h\gamma_2 b_4$ for some $b_3, b_4 \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$. This implies that $B_H(y) \subseteq B_H(b_3\gamma_1 h\gamma_2 b_4)$. Hence $y \leq_b b_3\gamma_1 h\gamma_2 b_4$. Since $x \leq y$ for some $y \in B_H(b_3\gamma_1 h\gamma_2 b_4)$, we have $x \in B_H(b_3\gamma_1 h\gamma_2 b_4)$. We will show that $B_H(x) \subseteq B_H(b_3\gamma_1 h\gamma_2 b_4)$. Consider

$$\begin{aligned} x \cup x\Gamma x \cup x\Gamma H\Gamma x &\subseteq B_H(b_3\gamma_1 h\gamma_2 b_4) \cup B_H(b_3\gamma_1 h\gamma_2 b_4)\Gamma B_H(b_3\gamma_1 h\gamma_2 b_4) \cup B_H(b_3\gamma_1 h\gamma_2 b_4)\Gamma H\Gamma B_H(b_3\gamma_1 h\gamma_2 b_4) \\ &\subseteq (b_3\gamma_1 h\gamma_2 b_4 \cup b_3\gamma_1 h\gamma_2 b_4\Gamma b_3\gamma_1 h\gamma_2 b_4 \cup b_3\gamma_1 h\gamma_2 b_4\Gamma H\Gamma b_3\gamma_1 h\gamma_2 b_4] \\ &= B_H(b_3\gamma_1 h\gamma_2 b_4). \end{aligned}$$

Then we have $B_H(x) = (x \cup x\Gamma x \cup x\Gamma H\Gamma x]$

$$\subseteq (B_H(b_3\gamma_1 h\gamma_2 b_4)] = B_H(b_3\gamma_1 h\gamma_2 b_4).$$

This implies that $B_H(x) \subseteq B_H(b_3\gamma_1 h\gamma_2 b_4)$. Hence $x \leq_b b_3\gamma_1 h\gamma_2 b_4$. The validity of (2) and (3) follows from Lemma 2.9(1) and Lemma 2.9(2) respectively.

Conversely, assume that (1), (2) and (3) are hold. We will show that B is a bi-base of H. To show that $B_H(B) = H$. Clearly $B_H(B) \subseteq H$. By (1) $H \subseteq B_H(B)$. Hence $H = B_H(B)$. It remains to show that B is a minimal subset of H, with the property $H = B_H(B)$. Suppose that $H = B_H(A)$ for some $A \subset B$. Since $A \subset B$, there exists $b \in B \setminus A$. Since $b \in B \subseteq H = B_H(A)$ and $b \notin A$, it follows that $b \in (A\Gamma A \cup A\Gamma H\Gamma A]$. So we have $b \leq y$ for some $y \in A\Gamma A \cup A\Gamma H\Gamma A$. We can consider the two following cases.

Case 1: $y \in A\Gamma A$. Then $y \in a_1\gamma a_2$ for some $a_1, a_2 \in A$ and $\gamma \in \Gamma$. Then $a_1, a_2 \in B$. Since $b \notin A$, we have $b \neq a_1$ and $b \neq a_2$. Since $y \in a_1\gamma a_2$, We have $y \leq_b a_1\gamma a_2$. Since $b \leq y$ for some $y \in B_H(a_1\gamma a_2)$, we have $b \in B_H(a_1\gamma a_2)$. We will show that $B_H(b) \subseteq B_H(a_1\gamma a_2)$. Consider

$$\begin{aligned} b \cup b\Gamma b \cup b\Gamma H\Gamma b &\subseteq B_H(a_1\gamma a_2) \cup B_H(a_1\gamma a_2)\Gamma \\ &\quad B_H(a_1\gamma a_2) \cup B_H(a_1\gamma a_2)\Gamma H\Gamma \\ &\quad B_H(a_1\gamma a_2) \\ &\subseteq (a_1\gamma a_2 \cup a_1\gamma a_2\Gamma a_1\gamma a_2 \cup a_1\gamma a_2 \\ &\quad \Gamma H\Gamma a_1\gamma a_2] \\ &= B_H(a_1\gamma a_2). \end{aligned}$$

Then we have $B_H(b) = (b \cup b\Gamma b \cup b\Gamma H\Gamma b]$

$$\subseteq (B_H(a_1\gamma a_2)] = B_H(a_1\gamma a_2).$$

This implies that $B_H(b) \subseteq B_H(a_1\gamma a_2)$. Hence $b \leq_b a_1\gamma a_2$. This contradicts to (2).

Case 2: $y \in A\Gamma H\Gamma A$. Then $y \in a_3\gamma_1 h\gamma_2 a_4$ for some $a_3, a_4 \in A$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$. Since $b \notin A$, we have $b \neq a_3$ and $b \neq a_4$. Since $A \subseteq B$, $a_3, a_4 \in B$. Since $y \in a_3\gamma_1 h\gamma_2 a_4$, so $B_H(y) \subseteq B_H(a_3\gamma_1 h\gamma_2 a_4)$. Hence, $y \leq_b a_3\gamma_1 h\gamma_2 a_4$. Since $b \leq y$ for some $y \in B_H(a_3\gamma_1 h\gamma_2 a_4)$, we have $b \in B_H(a_3\gamma_1 h\gamma_2 a_4)$. We will show that $B_H(b) \subseteq B_H(a_3\gamma_1 h\gamma_2 a_4)$. Consider

$$\begin{aligned} b \cup b\Gamma b \cup b\Gamma H\Gamma b &\subseteq B_H(a_3\gamma_1 h\gamma_2 a_4) \cup B_H(a_3\gamma_1 h\gamma_2 a_4) \\ &\quad \Gamma B_H(a_3\gamma_1 h\gamma_2 a_4) \cup B_H(a_3\gamma_1 h\gamma_2 a_4) \\ &\quad \Gamma H\Gamma B_H(a_3\gamma_1 h\gamma_2 a_4) \\ &\subseteq (a_3\gamma_1 h\gamma_2 a_4 \cup a_3\gamma_1 h\gamma_2 a_4\Gamma a_3\gamma_1 h\gamma_2 a_4 \\ &\quad \cup a_3\gamma_1 h\gamma_2 a_4\Gamma H\Gamma a_3\gamma_1 h\gamma_2 a_4] \\ &= B_H(a_3\gamma_1 h\gamma_2 a_4). \end{aligned}$$

Then we have $B_H(b) = (b \cup b\Gamma b \cup b\Gamma H\Gamma b]$

$$\subseteq (B_H(a_3\gamma_1 h\gamma_2 a_4)] = B_H(a_3\gamma_1 h\gamma_2 a_4).$$

This implies that $B_H(b) \subseteq B_H(a_3\gamma_1 h\gamma_2 a_4)$. Hence $b \leq_b a_3\gamma_1 h\gamma_2 a_4$. This contradicts to (3). Therefore B is a bi-base of H.

Theorem 2.11 *Let B be a bi-base of an ordered Γ-semihypergroup H. Then B is a sub Γ-semihypergroup of H if and only if B satisfies the conditions $b \in b\beta c$ or $c \in b\beta c$, for any $b, c \in B$ and $\beta \in \Gamma$.*

Proof. ($\Rightarrow$) Assume that B is a sub Γ-semihypergroup of H. Suppose that $b \notin b\beta c$ and $c \notin b\beta c$. Let $a \in b\beta c$. Thus $a \neq b$ and $a \neq c$. Consider $a \in b\beta c \subseteq (b\beta c \cup b\beta c\Gamma b\beta c \cup b\beta c\Gamma H\Gamma b\beta c]$. By Lemma 2.8(1) we have $a = b$ or $a = c$. This is contradiction.

($\Leftarrow$) Assume that $b \in b\beta c$ or $c \in b\beta c$ for any $b, c \in B$. Let $a \in B\Gamma B$. Thus $a \in b\beta c$ for some $b, c \in B$. Since $a \in (b\beta c \cup b\beta c\Gamma b\beta c \cup b\beta c\Gamma H\Gamma b\beta c]$, by Lemma 2.8(1) we have $a = b$ or $a = c$. Hence $a \in \{b, c\} \subseteq B$. Therefore B is a sub Γ-semihypergroup of H.

3. Conclusions

In this reseach, we introduced and studied some properties of bi-bases of ordered Γ-semihypergroups. We proved that a non-empty subset B of an ordered Γ-semihypergroup of H is a bi-base of H if and only if B satisfies the two following conditions (1) $H = B_H(B)$ ($i.e. H = (B \cup B\Gamma B \cup B\Gamma H\Gamma B)$; (2) if A a subset of B such that $H = B_H(A)$ then $A = B$. Also we prove that let B be a non-empty subset of an ordered Γ-semihypergroup H. If B is a bi-base of H, then for any $x \in H$ there exists $a \in B$ such that $x \leq_b a$. and for any two distinct elements $a, b \in B$ such that $a \neq b$, then neither $a \leq_b b$, nor $b \leq_b a$. Finally, Let B be a bi-base of an ordered Γ-semihypergroup H. Then B is a sub Γ-semihypergroup of H if and only if B satisfies the conditions $b \in b\beta c$ or $c \in b\beta c$, for any $b, c \in B$ and $\beta \in \Gamma$.

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