Discuss the motion of a particle in a central inverse-square-law force field for a superimposed force whose magnitude is inversely proportional to the cube of the distance from the particle to the force center; that is,

$$F(r) = -\frac{k}{r^2} - \frac{\lambda}{r^3} \qquad k, \lambda > 0$$

Show that the motion is described by a precessing ellipse. Consider the cases $\lambda < l^2/\mu$, $\lambda = l^2/\mu$, and $\lambda > l^2/\mu$.

Find the force law for a central-force field that allows a particle to move in a spiral orbit given by $r = k\theta^2$, where k is a constant.

Discuss the motion of a particle moving in an attractive central-force field described by $F(r) = -k/r^3$.* Sketch some of the orbits for different values of the total energy. Can a circular orbit be stable in such a force field?

A communications satellite is in a circular orbit around Earth at radius R and velocity v. A rocket accidentally fires quite suddenly, giving the rocket an outward radial velocity v in addition to its original velocity.

(a) Calculate the ratio of the new energy and angular momentum to the old.

(b) Describe the subsequent motion of the satellite and plot $T(r)$, $V(r)$, $U(r)$, and $E(r)$ after the rocket fires.

(a) By the virial theorem, $T = -U/2$ for a circular orbit.

The firing of the rocket doesn't change U, so $U_f = U_i$

But

$$T_f = \frac{1}{2}m\left(v^2 + v^2\right) = 2T_i$$

So

$$E_f = 2T_i + U_i = -U_i + U_i = 0$$

$$\boxed{\frac{E_f}{E_i} = 0}$$

The firing of the rocket doesn't change the angular momentum since it fires in a radial direction.

$$\boxed{\frac{\ell_f}{\ell_i} = 1}$$

(b) $E = 0$ means the orbit is parabolic. The satellite will be lost.

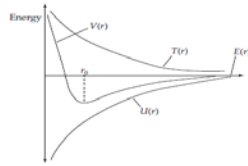
$$E(r) = 0 \qquad U(r) = -\frac{GM_e m_s}{r}$$

$$T(r) = E - U = \frac{GM_e m_s}{r}$$

$$V(r) = U(r) + \frac{\ell^2}{2\mu r^2} = -\frac{GM_e m_s}{r} + \frac{\ell^2}{2\mu r^2}$$

Behavior of $V(r)$ is determined by

$$\begin{bmatrix} \ell^2/2\mu r^2 & \text{for small } r \\ -GM_e m_s/r & \text{for large } r \end{bmatrix}$$



Minimum in $V(r)$ is found by setting $\frac{dV}{dr} = 0$ at $r = r_0$

$$0 = -\frac{GM_e m_s}{r_0^2} + \frac{\ell^2}{\mu r_0^3}$$

$$r_0 = -\frac{\ell^2}{\mu GM_e m_s}$$

Find the force law for a central-force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where k and α are constants.

Solution. We use Equation 8.21 to determine the force law $F(r)$. First, we determine

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{e^{-\alpha\theta}}{k}\right) = \frac{-\alpha e^{-\alpha\theta}}{k}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{\alpha^2 e^{-\alpha\theta}}{k} = \frac{\alpha^2}{r}$$

From Equation 8.21, we now determine $F(r)$.

$$F(r) = \frac{-l^2}{\mu r^2}\left(\frac{\alpha^2}{r} + \frac{1}{r}\right)$$

$$F(r) = \frac{-l^2}{\mu r^3}(\alpha^2 + 1) \qquad \textbf{(8.22)}$$

Determine $r(t)$ and $\theta(t)$ for the problem in Example 8.1.

Solution. From Equation 8.10, we find

$$\dot{\theta} = \frac{l}{\mu r^2} = \frac{l}{\mu k^2 e^{2\alpha\theta}}$$

Rearranging Equation 8.23 gives

$$e^{2\alpha\theta}d\theta = \frac{l}{\mu k^2}dt$$

and integrating gives

$$\frac{e^{2\alpha\theta}}{2\alpha} = \frac{lt}{\mu k^2} + C'$$

where C' is an integration constant. Multiplying by 2α and letting $C = 2\alpha C'$ gives

$$e^{2\alpha\theta} = \frac{2\alpha lt}{\mu k^2} + C \qquad (8.24)$$

We solve for $\theta(t)$ by taking the natural logarithm of Equation 8.24:

$$\theta(t) = \frac{1}{2\alpha}\ln\left(\frac{2\alpha lt}{\mu k^2} + C\right) \qquad (8.25)$$

We can similarly solve for $r(t)$ by examining Equations 8.23 and 8.24:

$$\frac{r^2}{k^2} = e^{2\alpha\theta} = \frac{2\alpha lt}{\mu k^2} + C$$

$$r(t) = \left[\frac{2\alpha l}{\mu}t + k^2C\right]^{1/2} \qquad (8.26)$$

What is the total energy of the orbit of the previous two examples?

Solution. The energy is found from Equation 8.14. In particular, we need $\dot{r}$ and $U(r)$.

$$U(r) = -\int F\,dr = \frac{+l^2}{\mu}(\alpha^2 + 1)\int r^{-3}dr$$

$$U(r) = -\frac{l^2(\alpha^2 + 1)}{2\mu}\frac{1}{r^2} \qquad \textbf{(8.}$$

where we have let $U(\infty) = 0$.

We rewrite Equation 8.10 to determine $\dot{r}$:

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d\theta}{dr}\frac{dr}{dt} = \frac{1}{\mu r^2}$$

$$\dot{r} = \frac{dr}{d\theta}\frac{l}{\mu r^2} = \alpha k e^{\alpha\theta}\frac{l}{\mu r^2} = \frac{\alpha l}{\mu r} \qquad \textbf{(8.}$$

Substituting Equations 8.27 and 8.28 into Equation 8.14 gives

$$E = \frac{1}{2}\mu\left(\frac{\alpha l}{\mu r}\right)^2 + \frac{l^2}{2\mu r^2} - \frac{l^2(\alpha^2 + 1)}{2\mu r^2}$$

$$E = 0$$

The total energy of the orbit is zero if $U(r = \infty) = 0$.

Determine whether a particle moving on the inside surface of a cone under the influence of gravity (see Example 7.4) can have a stable circular orbit.

Solution. In Example 7.4, we found that the angular momentum about the z-axis was a constant of the motion:

$$l = mr^2\dot{\theta} = \text{constant}$$

We also found the equation of motion for the coordinate r:

$$\ddot{r} - r\dot{\theta}^2\sin^2\alpha + g\sin\alpha\cos\alpha = 0 \tag{8.98}$$

If the initial conditions are appropriately selected, the particle can move in a circular orbit about the vertical axis with the plane of the orbit at a constant height z_0 above the horizontal plane passing through the apex of the cone. Although this problem does not involve a central force, certain aspects of the motion are the same as for the central-force case. Thus we may discuss, for example, the stability of circular orbits for the particle. To do this, we perform a perturbation calculation.

First, we assume that a circular orbit exists for $r = \rho$. Then, we apply the perturbation $r \to \rho + x$. The quantity $r\dot{\theta}^2$ in Equation 8.98 can be expressed as

$$\begin{aligned} r\dot{\theta}^2 &= r\cdot\frac{l^2}{m^2r^4} = \frac{l^2}{m^2r^3} \\ &= \frac{l^2}{m^2}(\rho + x)^{-3} = \frac{l^2}{m^2\rho^3}\left(1 + \frac{x}{\rho}\right)^{-3} \\ &\cong \frac{l^2}{m^2\rho^3}\left(1 - 3\frac{x}{\rho}\right) \end{aligned}$$

where we have retained only the first term in the expansion, because x/ρ is by hypothesis a small quantity.

Then, because $\ddot{\rho} = 0$, Equation 8.98 becomes, approximately,

$$\ddot{x} - \frac{l^2\sin^2\alpha}{m^2\rho^3}\left(1 - 3\frac{x}{\rho}\right) + g\sin\alpha\cos\alpha = 0$$

or

$$\ddot{x} + \left(\frac{3l^2\sin^2\alpha}{m^2\rho^4}\right)x - \frac{l^2\sin\alpha}{m^2\rho^3} + g\sin\alpha\cos\alpha = 0 \tag{8.99}$$

If we evaluate Equation 8.98 at $r = \rho$, then $\ddot{r} = 0$, and we have

$$\begin{aligned} g\sin\alpha\cos\alpha &= \rho\dot{\theta}^2\sin^2\alpha \\ &= \frac{l^2}{m^2\rho^3}\sin^2\alpha \end{aligned}$$

In view of this result, the last two terms in Equation 8.99 cancel, and there remains

$$\ddot{x} + \left(\frac{3l^2\sin^2\alpha}{m^2\rho^4}\right)x = 0$$

The solution to this equation is just a harmonic oscillation with a frequency ω, where

$$\omega = \frac{\sqrt{3}l}{m\rho^2}\sin\alpha \tag{8.101}$$

Thus, the circular orbit is stable.

พิจารณา central force ที่อยู่ในรูป $F(r) = -\frac{k}{r^n}$ จงหาเงื่อนไขของ n ที่ทำให้เกิด stable circular orbit

__วิธีทำ__ ในขั้นต้น เราทำการ integrate เพื่อคำนวณฟังก์ชันของพลังงานศักย์ $U(r)$ โดยอาศัยคำนิยาม $-\frac{d}{dr}U(r) = F(r)$ จะได้ว่า

$$\begin{aligned} \int dU &= \int F(r)dr \\ U(r) &= -\frac{k}{n-1}\frac{1}{r^{n-1}} + C \end{aligned}$$

ถ้ากำหนดให้ $U(\infty) = 0$ แสดงว่า ค่าคงที่ของการ integrate $C = 0$ ดังนั้น $U(r) = -\frac{k}{n-1}\frac{1}{r^{n-1}}$ ส่งผลให้ effective potential อยู่ในรูปของ

$$U_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} - \frac{k}{n-1}\frac{1}{r^{n-1}}$$

จากสมการ (3.18) เราเริ่มด้วยการหาตำแหน่งของรัศมี R ที่ทำให้เกิดสภาวะ equilibrium กล่าวคือ

$$\left.\frac{d}{dr}U_{\text{eff}}(r)\right|_{r=R} = 0 = \left.\left(-\frac{\ell^2}{\mu r^3} + \frac{k}{r^n}\right)\right|_{r=R} = -\frac{\ell^2}{\mu R^3} + \frac{k}{R^n} = 0$$

เมื่อแก้สมการหาผลเฉลย R จะได้ว่า

$$R^{n-3} = \frac{\mu k}{\ell^2}$$

ในลำดับสุดท้ายคือการหาอนุพันธ์อันดับสองเทียบกับ r ณ ตำแหน่ง $r = R$ ซึ่งก็คือ

$$\left.\frac{d^2}{dr^2}U_{\text{eff}}(r)\right|_{r=R} = \left.\left(\frac{3\ell^2}{\mu r^4} - \frac{nk}{r^{n+1}}\right)\right|_{r=R} = \frac{3\ell^2}{\mu R^4} - \frac{nk}{R^{n+1}} =$$

จากสมการ (3.18) stable circular orbital จะเกิดได้ ก็ต่อเมื่อ $\left.\frac{d^2}{dr^2}U_{\text{eff}}(r)\right|_{r=R} > 0$ หรือ

$$\begin{aligned} \frac{3\ell^2}{\mu R^4} - \frac{nk}{R^{n+1}} &> 0 \\ \frac{3\ell^2}{\mu} - \frac{nk}{R^{n-3}} &> 0 \end{aligned}$$

แทน $R^{n-3} = \frac{\mu k}{\ell^2}$ ลงในอสมการข้างต้น จะได้ว่า $(3-n)\frac{\ell^2}{\mu} > 0$ เพราะฉะนั้น

stable circular orbit condition $\boxed{n < 3}$ __ตอบ__

Investigate the stability of circular orbits in a force field described by the potential function

$$U(r) = \frac{-k}{r}e^{-(r/a)}$$

where $k > 0$ and $a > 0$.

Solution. This potential is called the **screened Coulomb potential** (when $k = Ze^2/4\pi\varepsilon_0$, where Z is the atomic number and e is the electron charge)

$$F(r) = -\frac{\partial U}{\partial r} = -k\left(\frac{1}{ar} + \frac{1}{r^2}\right)e^{-(r/a)}$$

and

$$\frac{\partial F}{\partial r} = k\left(\frac{1}{a^2r} + \frac{2}{ar^2} + \frac{2}{r^3}\right)e^{-(r/a)}$$

The condition for stability (see Equation 8.93) is

$$3 + \rho\frac{F'(\rho)}{F(\rho)} > 0$$

Therefore

$$3 + \frac{\rho k\left(\frac{1}{a^2\rho} + \frac{2}{a\rho^2} + \frac{2}{\rho^3}\right)}{-k\left(\frac{1}{a\rho} + \frac{1}{\rho^2}\right)} > 0$$

which simplifies to

$$a^2 + a\rho - \rho^2 > 0$$

We may write this as

$$\frac{a^2}{\rho^2} + \frac{a}{\rho} - 1 > 0$$

Stability thus results for all $q \equiv a/\rho$ that exceed the value satisfying the equation

$$q^2 + q - 1 = 0$$

The positive (and therefore the only physically meaningful) solution is

$$q = \frac{1}{2}(\sqrt{5} - 1) \cong 0.62$$

If, then, the angular momentum and energy allow a circular orbit at $r = \rho$, the motion is stable if

$$\frac{a}{\rho} \gtrsim 0.62$$

or

$$\rho \lesssim 1.62a \tag{8.97}$$

ระบบ 2 อนุภาคซึ่งมี reduced mass เท่ากับ μ ซึ่งกำลังเคลื่อนที่ภายใต้แรง $F(r) = -\frac{\ell^2}{\mu r^3}\left(\alpha^2 + 1\right)$ เมื่อ k และ α คือ constants จงคำนวณหา total energy ของระบบดังกล่าว

__วิธีทำ__ จากสมการ (3.16) เราทำการเปลี่ยนข้อมูลของแรง $F(r) = -\frac{\ell^2}{\mu r^3}\left(\alpha^2 + 1\right)$ ให้เป็นข้อมูลของพลังงานศักย์ $U(r)$ โดยอาศัยสมการความสัมพันธ์ $-\frac{d}{dr}U(r) = F(r)$ ทั้งนี้เมื่อ integrate

$$\begin{aligned} \int dU &= \int F(r)dr \\ U(r) &= -\frac{\ell^2\left(\alpha^2 + 1\right)}{2\mu}\frac{1}{r^2} + C \end{aligned}$$

ค่าคงที่ C ของการ integrate สามารถหาได้จากการกำหนดให้ พลังงานศักย์ ณ ตำแหน่ง $r \to \infty$ มีค่าเป็นศูนย์ หรือ $U(\infty) = 0$ เพราะฉะนั้น $C = 0$ ดังนั้นเราจะได้ว่า ระบบมีพลังงานศักย์คือ

$$U(r) = -\frac{\ell^2\left(\alpha^2 + 1\right)}{2\mu}\frac{1}{r^2}$$

จากนั้นแทน $U(r)$ ในสมการ (3.16) ทำให้

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{\ell^2\left(\alpha^2 + 1\right)}{2\mu}\frac{1}{r^2} = \frac{1}{2}\mu\dot{r}^2 + \cancel{\frac{\ell^2}{2\mu r^2}} - \frac{\ell^2\alpha^2}{2\mu r^2} - \cancel{\frac{\ell^2}{2\mu r^2}}$$

เพราะฉะนั้นแล้ว พลังงาน $\boxed{E = \frac{1}{2}\mu\dot{r}^2 - \frac{\ell^2\alpha^2}{2\mu r^2}}$ __ตอบ__