On the Diophantine Equation $8^x + n^y = z^2$

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Abstract: - Let n be an positive integer with $n \equiv 10 (\bmod 15)$. In this paper, we prove that $(1,0,3)$ is unique non-negative integer solution (x,y,z) of the Diophantine equation $8^x + n^y = z^2$, where x,y and z are non-negative integers.

Key-Words: exponential Diophantine equation, Mersenne primes, solution, Factor, positive integral, non-negative integer

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1 Introduction

In 2012, Sroysang proved that $(1,0,3)$ is unique solution (x,y,z) for the Diophantine equation $8^x + 19^y = z^2$ [1]. In 2014, Sroysang also showed that $(1,0,3)$ is a unique the solution (x,y,z) for Diophantine equation $8^x + 13^y = z^2$ where x,y and z are non-negative integers. [2]. Moreover, he proved that $(1,0,3)$ is a unique non-negative integer solution (x,y,z) for the Diophantine equation $8^x + 59^y = z^2$ where x,y and z are non-negative integers [3]. In 2015, L an Qi and Xiaoxue Li showed that the Diophantine equation $8^x + p^y = z^2$ if $p \equiv \pm 3 (mod\ 8)$ has no non -negative solutions (x,y,z), if $p \equiv 7 (mod\ 8)$, is a unique solutions $(p,x,y,z) = (2^q - 1, (1/3)(q+2), 2, 2^q + 1)$, where q is an odd prime with $q \equiv 1 (mod\ 3)$; if $p \equiv 1 (mod\ 8)$ and $p \neq 17$, then the equation has at most two positive integer solutions (x,y,z) [4]. In 2017, Asthana have shown that the Diophantine equation $8^x + 113^y = z^2$ has only three non-negative integer solutions where x, y and z are non-negative integers. The solutions (x,y,z) are $(1, 0, 3)$, $(1, 1, 11)$ and $(3, 1, 25)$ [5]. In 2019, Makate N., Srimud K., Warong A. and Supjaroen W. showed that the two Diophantine equations $8^x + 61^y = z^2$ and $8^x + 67^y = z^2$ have a unique solution, that is $(x,y,z) = (1,0,3)$ [6]. In the same year Burshtein established in a very elementary manner that the equation $8^x + 9^y = z^2$ has no s olutions when x,y and z are positive integers. These results are achieved in particular by utilizing the last digits of the powers $8^x, 9^y$ [7]. In 2020, A. Elshahed A. and Kamarulhaili H. have shown that the Diophantine equation $(4^n)^x - p^y = z^2$, where p is an odd prime, $n \in \mathrm{Z}^+$ and x,y,z are non-negative integers, has been investigated to show that the solutions are given by $\{(x,y,z,p)\} = \{(k, 1, 2\ nk - 1, 2\ nk + 1 - 1)\} \cup \{(0,\ 0,\ 0,\ p)\}$ [8].

In this paper we consider some Diophantine equations $8^x + n^y = z^2$ where n be an positive integer with $n \equiv 10 (\bmod 15)$, x, y and z are non-negative integers.

2 Preliminaries

Let $n \equiv 10 (\bmod 15)$. In this paper we assume that n is a non-negative integer. It is clear that $n \equiv 1 (\bmod 3)$ and $n \equiv 0 (\bmod 5)$

Lemma 1. $(a,b,x,y)=(3,2,2,3)$ is a unique solution of the Diophantine equation $a^x - b^y = 1$ where a,b,x and y are integers with $\min\{a,b,x,y\}>1$

Proof see Mihailescu [9]

Lemma 2 $(1,3)$ is a unique solution (x,z) for the Diophantine equation $8^x + 1 = z^2$ where x and z are non-negative integers.

Proof see Sroysang [1]

Let p be an odd prime and a be a positive integer where $gcd(a,p)=1$. If the quadratic congruence $x^2 \equiv a \pmod p$ has a solution, then a is said to *be a quadratic residue of* p. Otherwise, a is called *a quadratic non-residue* of p. In 1798 Adrien-Marie Legendre introduced the Legendre symbol $\left(\frac{a}{p}\right)$ which is defined by

$$\left(\frac{a}{p}\right)=\begin{cases}1 & ;\ \text{if } a \text{ is a quadratic residue of } p,\\ \text{-1} & ;\ \text{if } a \text{ is a quadratic non-residue of } p.\end{cases}$$

In this paper, using following these symbols.

Theorem 3. If p is an odd prime, then [10]

$$\left(\frac{2}{p}\right)=\begin{cases}1 & ;\text{if } p\equiv 1 \pmod 8 \text{ or } p\equiv 7 \pmod 8\\ -1 & ;\text{if } p\equiv 3 \pmod 8 \text{ or } p\equiv 5 \pmod 8\end{cases}$$

Theorem 4. If $p \neq 3$ is an odd prime, then [10]

$$\left(\frac{3}{p}\right)=\begin{cases}1 & ;\text{if } p\equiv \pm 1 \pmod{12}\\ -1 & ;\text{if } p\equiv \pm 5 \pmod{12}\end{cases}$$

Lemma 5. Let n be an positive integer with $n \equiv 10 \pmod{15}$.The Diophantine equation $1+n^y = z^2$ has non-negative integer solution y and z are non-negative integers.

Proof Let n be an positive integer with $n \equiv 5 \pmod{20}$, y and z. are non-negative integers. Then we consider three cases.

Case 1 if $y=0$. Then $2 = z^2$, is not possible.

Case 2 if $y=1$. Then $z^2 - 1 = n$, we get $z^2 - 1 \equiv 1 \pmod 3$ and $z^2 - 1 \equiv 0 \pmod 5$ or $z^2 \equiv 2 \pmod 3$ and $z^2 \equiv 1 \pmod 5$. That is $\left(\frac{2}{3}\right) = -1$. By Theorem 3. In this case, there is no non-negative integer solution.

Case 3 if $y>1$. Then $z^2 = 1 + n^y > 11$. This implies $z>3$. Here $\min\{y,z\}>1$, by Lemma 1, this equation has no solution.

3 Main theorem

Theorem 6. Let n be an positive integer $n \equiv 10 \pmod{15}$. $(1,0,3)$ is a unique solution (x,y,z) of the Diophantine equation $8^x + n^y = z^2$, where x,y and z are non-negative

Proof Let n be an positive integer $n \equiv 10 \pmod{15}$, and x,y and z are non-negative integers .

Then there three cases by the following:

Case 1 if $x=0$. By Lemma 5, there is no non-negative integers solution.

Case 2 if $x \geq 1$. and $y=0$. By Lemma 2, we have $x=1$ and $z=3$.

Case 3 if $x \geq 1$ and $y \geq 1$. Then we consider two cases.

Case 3.1 x is odd, we get $8^x \equiv 2 \pmod 5$ or $8^x \equiv 3 \pmod 5$. Therefor $z^2 = 8^x + n^y \equiv 3,2 \pmod 5$.

That is $\left(\frac{2}{5}\right)=1$ and $\left(\frac{3}{5}\right)=1$. This is contradiction to Theorem 3 and Theorem 4, respectively. In this case, there is no non-negative integer solution.

Case 3.2 x is even, we get $8^x \equiv 1 \pmod 3$. Therefor $z^2 = 8^x + n^y \equiv 2 \pmod 3$ That is $\left(\frac{2}{3}\right)=1$ This is contradiction to Theorem 3. In this case, there is no non-negative integer solution.

Therefore $(1,0,3)$ is unique solution (x,y,z) for the equation $8^x + n^y = z^2$ where x,y and z are non-negative integers.

Example 7 $(1,0,3)$ is a unique solution (x,y,z) for the Diophantine equation $8^x + 10^y = z^2$, where x,y and z are non-negative integers.

Since $10 \equiv 10 \pmod{15}$, therefor by Theorem 6 $(1,0,3)$ is a unique solution (x,y,z) for the Diophantine equation $8^x + 10^y = z^2$, where x,y and z are non-negative integers.

Example 8 $(1,0,3)$ is a unique solution (x,y,z) for the Diophantine equation $8^x + 175^y = z^2$, where x, y and z are non-negative integers.

Since $175 \equiv 10 (\bmod 15)$, therefor by Theorem 6 $(1,0,3)$ is a unique solution (x,y,z) for the Diophantine equation $8^x + 175^y = z^2$, where x, y and z are non-negative integers.

Corollary 9 $(1,0,3)$ is a unique solution (x,y,z) for the Diophantine equation $8^x + 25^y = k^{4t+6}$, where x, y and z are non-negative integers.

Proof :. Let $k^{2t+3} = z$, for k and t are positive integer and $25 \equiv 10 (\bmod 15)$ then Diophantine equation becomes $8^x + 25^y = z^2$, therefor by Theorem 6 $(1,0,3)$ is a unique solution (x,y,z) for the Diophantine equation $8^x + 25^y = k^{4t+6}$, where x, y and z are non-negative integers.

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