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Sixteenth-Order Iterative Method for Solving Nonlinear Equations

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Abstract

In this paper, we suggest and analyze some new sixteen-order iterative methods by using Householder's method free from second derivative for solving nonlinear equations. Here we use a new and different technique for implementation of sixteen-order derivative of the function. The efficiency index equals $16^{\frac{1}{6}} \approx 1.587$. Numerical examples of the new methods are compared with other methods by exhibiting the effectiveness of the method presented in this paper.

1 Introduction

A common problem in engineering, scientific computing and applied mathematics, in general, is the problem of solving a nonlinear equation $f(x) = 0$.

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To find a zero of the non-linear equation, Newton's method [14] is one of the well known optimal methods using:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1.1}$$

There exists numerous modifications of the Newton's method which improve the convergence rate (see [1, 5, 7, 8, 9, 10, 12, 15, 16, 17] and references therein). For the sake of completeness, we list some existing optimal sixteenth-order convergent methods. In 2011, Geum and Kim [2] proposed a biparametric family of optimally convergent sixteenth-order multipoint methods (GE1):

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= -K_f \frac{f(y_n)}{f'(x_n)} \\ s_n &= z_n - H_f \frac{f(z_n)}{f'(x_n)} \\ x_{n+1} &= s_n - W_f \frac{f(s_n)}{f'(x_n)}, \end{aligned} \tag{1.2}$$

where $u_n = \frac{f(y_n)}{f(x_n)}$, $v_n = \frac{f(z_n)}{f(y_n)}$, $w_n = \frac{f(z_n)}{f(x_n)}$, $t_n = \frac{f(s_n)}{f(z_n)}$, $K_f = \frac{1+\beta u_n+(-9+5/2\beta)u_n^2}{1+(\beta-2)u_n+(-4+\beta/2)u_n^2}$, $H_f = \frac{1+2u_n+(2+\sigma)w_n}{1-v_n+\sigma w_n}$, $W_f = \frac{1+2u_n+(2+\sigma)v_n w_n}{1-v_n-2w_n-t_n+2(1+\sigma)v_n w_n} + G$
one of the choices for G along with $\beta = 2$ and $\sigma = -2$:
$G = -\frac{1}{2}\left[u_n w_n(6+12u_n+(24-11\beta)u_n^2+u_n^3 p_1+4\sigma)\right] + p_2 w_n^2$, $p_1 = (11\beta^2 - 66\beta + 136)$, $p_2 = (2u_n(\sigma^2 - 2\sigma - 9) - 4\sigma - 6)$

In the same year, Geum and Kim [3] presented a family of optimal sixteenth-order multipoint methods (GE2)

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= -K_f \frac{f(y_n)}{f'(x_n)} \\ s_n &= z_n - H_f \frac{f(z_n)}{f'(x_n)} \\ x_{n+1} &= s_n - W_f \frac{f(s_n)}{f'(x_n)}, \end{aligned} \tag{1.3}$$

where $u_n = \frac{f(y_n)}{f(x_n)}$, $v_n = \frac{f(z_n)}{f(y_n)}$, $w_n = \frac{f(z_n)}{f(x_n)}$, $t_n = \frac{f(s_n)}{f(z_n)}$, $K_f = \frac{1+\beta u_n+(-9+5/2\beta)u_n^2}{1+(\beta-2)u_n+(-4+\beta/2)u_n^2}$, $H_f = \frac{1+2u_n+(2+\sigma)w_n}{1-v_n+\sigma w_n}$, $W_f = \frac{1+2u_n}{1-v_n-2w_n-t_n} + G$
one of the choices for G along with $\beta = \frac{24}{11}$ and $\sigma = -2$: $G = -6u_n^3 v_n - \frac{244}{11}u_n^4 w_n + 6w_n^2 + u_n(2v_n^2 + 4v_n^3 + w_n - 2w_n^2)$

In 2012, Thukral [13] presented a four-point derivative-free sixteenth-order iterative methods (THU)

$$
\begin{aligned}
w_n &= x_n + f(x_n) \\
y_n &= x_n - \frac{f(x_n)}{f[x_n, y_n]} \\
z_n &= y_n - \phi \frac{f(y_n)}{f[x_n, y_n]} \\
a_n &= z_n - \eta \frac{f(z_n)}{f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n]} \\
x_{n+1} &= z_n - \sigma \frac{f[y_n, z_n] f(a_n)}{f[y_n, a_n] f[z_n, a_n]},
\end{aligned}
\tag{1.4}
$$

where $u_1 = \frac{f(z_n)}{f(x_n)}$, $u_2 = \frac{f(z_n)}{f(w_n)}$, $u_3 = \frac{f(y_n)}{f(x_n)}$, $u_4 = \frac{f(y_n)}{f(w_n)}$, $u_5 = \frac{f(a_n)}{f(x_n)}$, $u_6 = \frac{f(a_n)}{f(w_n)}$, $\phi = \frac{f[x_n, w_n]}{f[y_n, w_n]}$, $\eta = \frac{1}{(1+2u_3 u_n^2)(1-u_2)}$,
$\sigma = \frac{1+u_1 u_2 - u_1 u_3 u_4^2 + u_5 + u_6 + u_1^2 u_4 + u_2^2 u_3 + 3u_1 u_4^2 (u_3^2 - u_4^2)}{f[x_n, y_n]}$

In 2017, Rafiullah and Jabeen [11] proposed Sixteenth Order Iterative Methods (RAF)

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
z_n &= y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f(y_n)^2 (f'(x_n) - f'(y_n))}{2(f(x_n - f(y_n)) f'(x_n)^2} \\
v_n &= z_n - \frac{f(z_n)((x_n - y_n)(x_n - z_n)(y_n - z_n))}{-f(z_n)(x_n - y_n)(x_n - 2z_n + y_n) + f(y)(x_n - z_n)^2 - f(x_n)(y_n - z_n)^2} \\
x_{n+1} &= v_n - \frac{f(v_n)}{dfv},
\end{aligned}
\tag{1.5}
$$

where $dfv = \frac{f(v_n)}{v_n - x_n} + \frac{f(v_n)}{v_n - y_n} + \frac{f(v_n)}{v_n - z_n} + \frac{f(x_n)(v_n - y_n)(v_n - z_n)}{(x_n - v_n)(x_n - y_n)(x_n - z_n)} + \frac{f(y_n)(v_n - x_n)(v_n - z_n)}{(v_n - y_n)(x_n - y_n)(y_n - z_n)} + \frac{f(z_n)(v_n - x_n)(v_n - y_n)}{(z_n - v_n)(z_n - x_n)(z_n - y_n)}$

Our proposed iterative method was developed from a concept of Mylapalli, Palli and Vatti [10] and Householders method [4]. The proposed algorithms are applied to solve some test examples in order to assess its validity and accuracy.

2 Iterative Methods

Consider the nonlinear equation

$$f(x) = 0, \tag{2.6}$$

with x as a simple root, x_n an initial guess and ε is the error. Thus,

$$x = x_n + \varepsilon. \tag{2.7}$$

Using Taylor's formula, equation (2.6) can be written in the form of the following coupled system:

$$\begin{aligned} f(x) &= f(x_n) + (x - x_n)f'(x_n) + \frac{(x - x_n)^2}{2!}f''(x_n) + \dots \\ &= f(x_n) + f'(x_n)\varepsilon + \frac{f''(x_n)}{2!}\varepsilon^2 + \dots \end{aligned} \tag{2.8}$$

From equation (2.6) and (2.8), we get

$$f''(x_n)\varepsilon^2 + 2f'(x_n)\varepsilon + 2f(x_n) = 0. \tag{2.9}$$

We solve for ε to obtain

$$\varepsilon = \frac{-2f'(x_n) \pm \sqrt{(2f'(x_n))^2 - 8f(x_n)f''(x_n)}}{2f''(x_n)} \tag{2.10}$$

On Substituting x by x_{n+1} in (2.7) and from (2.10), we get

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)(1 + \sqrt{1 - 2p_n})}, \tag{2.11}$$

where $p_n = \frac{f(x_n)f''(x_n)}{(f'(x_n))^2}$.

Rewriting the above equation with Newton's method as a predictor gives us a new algorithm as follows:

Algorithm 2.1 For a given x_0, compute approximate solutions x_{n+1} by the iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{2f(y_n)}{f'(y_n)\left(1 + \sqrt{1 - 2p_n}\right)} \end{aligned} \tag{2.12}$$

where $p_n = \frac{f(y_n)f''(y_n)}{f'(y_n)^2}$.

In the next method, a step of iteration was added by Householders method, which has cubic convergence[4]. Thus the new iteration method is obtained as Algorithm 2.2:

Algorithm 2.2 For a given x_0, compute approximate solutions x_{n+1} by the iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{2f(y_n)}{f'(y_n)\left(1+\sqrt{1-2p_n}\right)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)} - \frac{f(z_n)f''(z_n)}{2f'^3(z_n)} \end{aligned} \tag{2.13}$$

In order to implement this method, one has to find the second derivative of this function, which may create some problems. To overcome this drawback, we use a new and different technique to reduce the second derivative of the function to the first derivative. This idea plays a significant role in developing some new iterative methods free from second derivatives. To be more precise, we consider

$$f''(y_n) = \frac{2}{y_n - x_n}\left(2f'(y_n) + f'(x_n) - 3\frac{f(y_n)-f(x_n)}{y_n - x_n}\right) \tag{2.14}$$

$$f''(z_n) = d_n = \frac{2}{z_n - y_n}\left(2f'(z_n) + f'(y_n) - 3\frac{f(z_n)-f(y_n)}{z_n - y_n}\right). \tag{2.15}$$

We suggest the following new iterative method for solving the nonlinear equation and this is the new motivation of higher-order.

Algorithm 2.3 For a given x_0, compute approximate solutions x_{n+1} by the iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{2f(y_n)}{f'(y_n)\left(1+\sqrt{1-2p_n}\right)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)} - \frac{f(z_n)d_n}{2f'^3(z_n)}. \end{aligned} \tag{2.16}$$

Algorithm 2.3 is a new three-step iteration method (TSI) with the sixteenth order convergence. Thus, Algorithm 2.3 efficiency index is $16^{\frac{1}{6}} \approx 1.5874$

3 Convergence Analysis

In this section, we examine a convergence analysis of the newly proposed algorithm in the form of the following theorem:

Theorem 3.1. *Suppose that α is a root of the equation $f(x) = 0$. If $f(x)$ is sufficiently smooth in the neighborhood of α, then the order of convergence of Algorithm 2.3 is sixteen.*

Proof. To analyze the convergence of Algorithm 2.3, suppose that α is a root of the equation $f(x) = 0$ and e_n is the error at the nth iteration. Then $e_n = x_n - \alpha$.By using a Taylor series expansion, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + \ldots] \quad (3.17)$$

$$f'(x_n) = f'(\alpha)[1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+6c_6e_n^5+7c_7e_n^6+\ldots], \quad (3.18)$$

where $c_n = \dfrac{f^{(n)}(\alpha)}{n!f'(\alpha)}$.

With the help of equations ((3.17)) and (3.18), we get

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} =& e_n - c_2e_2^2 - (2c_3 - 2c_2^2)e_n^3 - (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 \\ &+ 2(5c_2c_4 + 4c_2^4 - 10c_2^2c_3 + 3c_3^2 - 2c_5)e_n^5 + \ldots\end{aligned} \quad (3.19)$$

$$\begin{aligned}y_n =& \alpha + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 - (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 \\ &+ (-8c_2^4 + 20c_2^2c_3 - 10c_2c_4 - 6c_3^2 + 4c_5)e_n^5 + \ldots\end{aligned} \quad (3.20)$$

$$\begin{aligned}f(y_n) =& f'(\alpha)[c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &+ (-12c_2^4 + 24c_2^2c_3 - 10c_2c_4 - 6c_3^2 + 4c_5)e_n^5 + \ldots]\end{aligned} \quad (3.21)$$

$$\begin{aligned}f'(y_n) =& f'(\alpha)[1 + 2c_2^2e_n^2 + 4(c_3c_2 - c_2^3)e_n^3 + (6c_2c_4 - 11c_3c_2^2 + 8c_2^4)e_n^4 \\ &+ (-16c_2^5 + 28c_2^3c_3 - 20c_2^2c_4 + 8c_2c_5)e_n^5 + \ldots]\end{aligned} \quad (3.22)$$

Using equations (3.17)-(3.22), we get

$$z = \alpha + (-c_2^3c_3 + c_2^2c_4)e_n^6 + (6c_2^4c_3 - 6c_2^3c_4 - 6c_2^2c_3^2 + 2c_2^2c_5 + 4c_2c_3c_4)e_n^7 + \ldots \quad (3.23)$$

$$f(z_n) = f'(\alpha)[(-c_2^3c_3 + c_2^2c_4)e_n^6 + (6c_2^4c_3 - 6c_2^3c_4 - 6c_2^2c_3^2 + 2c_2^2c_5 + 4c_2c_3c_4)e_n^7 \ldots] \quad (3.24)$$

$$f'(z_n) = f'(\alpha)[1 + (-2c_2^4c_3 + 2c_2^3c_4)e_n^6 + (12c_2^5c_3 - 12c_2^4c_4 - 12c_2^3c_3^2 + 4c_2^3c_5 + 8c_2^2c_3c_4)e_n^7 + \ldots] \quad (3.25)$$

Using equations (3.23)-(3.25), we get

$$x_{n+1} = \alpha + \left(c_2^8c_3^2c_4 - 2c_2^7c_3c_4^2 + c_2^6c_4^3\right) e_n^{16} + O(e_n^{17}), \quad (3.26)$$

which implies that

$$e_{n+1} = \left(c_2^8c_3^2c_4 - 2c_2^7c_3c_4^2 + c_2^6c_4^3\right) e_n^{16} + O(e_n^{17}). \quad (3.27)$$

The above equation shows that the order of convergence of Algorithm 2.4 is sixteen. □

4 Numerical Experiments

In this section, we compare the number of iterations in obtaining an approximate root of our proposed methods with the other methods that have an equal order of convergence. Algorithm 2.3 (TSI) sixteenth order convergence compare with Geum and Kim (GE1) [2], Geum and Kim (GE2)[3], Thukral (THU) [13] and Rafiullah and Jabeen (RAF) [11]. We consider the following numerical examples:

$$f_1(x) = \sin(x) + \cos(x) + x,\ x_0 = -1.0$$
$$f_2(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,\ x_0 = -1.2$$
$$f_3(x) = (x+2)e^x - 1,\ x_0 = -0.9$$
$$f_4(x) = x^3 - 2x^2 - 5,\ x_0 = 2.0$$
$$f_5(x) = \cos(x) - x,\ x_0 = 1.7$$
$$f_6(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,\ x_0 = -1.0$$
$$f_7(x) = \sin^2(x) - x^2 + 1,\ x_0 = -2.5$$
$$f_8(x) = (x-1)e^{-x},\ x_0 = 0.25.$$

All examples were done using Maple with 3500 significant digits. The comparison was under the condition that the program will stop when $|x_n -$

$x_{n-1}| < \epsilon$ and $|f(x_n)| < \epsilon$, where $\epsilon = 10^{-200}$. Table 1 represents the number of iterations N, the approximate root x_{n+1}, the magnitude $|f(x)|$ of $f(x)$ at the final estimate x_{n+1}, the difference between two consecutive approximations $x_{n+1} - x_n$ of the equation and CPU time.

Table 1 Convergence for sample test functions $f_1(x) - f_8(x)$.

Method	**N**	x_n	$\lvert f(x_n)\rvert$	$\lvert x_n - x_{n-1}\rvert$	**time**
$f_1(x)$, $x_0 = -1$					
GE1	7	-0.456624704567630824437697457	4.57e-214	3.94e-107	0.042
GE2	2	-0.456624704567630824437697457	1.36e-236	4.51e-17	0.010
THU	5	-0.456624704567630824437697457	5.68e-516	2.10e-129	0.038
RAF	2	-0.456624704567630824437697457	3.50e-320	9.31e-23	0.012
TSI	2	-0.456624704567630824437697457	2.87e-385	6.58e-24	0.009
$f_2(x)$, $x_0 = -1.2$					
GE1	7	-1.207647827130918927009416758	1.49e-244	1.95e-123	0.057
GE2	2	-1.207647827130918927009416758	2.32e-338	2.72e-25	0.018
THU	4	-1.207647827130918927009416758	7.44e-266	3.82e-68	0.045
RAF	2	-1.207647827130918927009416758	7.66e-366	3.77e-27	0.020
TSI	2	-1.207647827130918927009416758	1.68e-521	1.98e-33	0.011
$f_3(x)$, $x_0 = -0.9$					
GE1	div	-	-	-	-
GE2	27	-0.442854401002388583141327999	5.30e-1078	1.08e-77	0.278
THU	8	-0.442854401002388583141327999	8.04e-238	3.50e-60	0.058
RAF	3	-0.442854401002388583141327999	1.60e-1464	4.96e-105	0.020
TSI	3	-0.442854401002388583141327999	4.95e-2216	5.90e-139	0.014
$f_4(x)$, $x_0 = 2$					
GE1	div	-	-	-	-
GE2	165	2.690647448028613750350788826	1.02e-419	1.09e-30	0.472
THU	6	2.690647448028613750350788826	2.43e-678	8.56e-171	0.021
RAF	4	2.690647448028613750350788826	1.09e-812	2.58e-58	0.008
TSI	3	2.690647448028613750350788826	2.69e-1897	7.58e-106	0.004
$f_5(x)$, $x_0 = 1.7$					
GE1	9	0.7390851332151606416553120876	1.29e-381	5.22e-191	0.192
GE2	3	0.7390851332151606416553120876	1.79e-1969	4.84e-141	0.060
THU	5	0.7390851332151606416553120876	1.65e-558	8.79e-140	0.118
RAF	3	0.7390851332151606416553120876	5.99e-2574	5.63e-184	0.086
TSI	2	0.7390851332151606416553120876	1.99e-242	3.46e-15	0.042
$f_6(x)$, $x_0 = -1.0$					

GE1	div	-	-	-	-
GE2	div	-	-	-	-
THU	6	-1.2076478271309189270094167758	2.16e-322	2.81e-82	0.373
RAF	div	-	-	-	-
TSI	3	-1.2076478271309189270094167758	1.81e-2475	1.49e-155	0.219
$f_7(x)$, $x_0 = -2.5$					
GE1	17	-1.4044916482153412260350868	4.58e-207	4.28e-104	0.534
GE2	3	-1.4044916482153412260350868	2.91e-856	5.60e-62	0.079
THU	6	-1.4044916482153412260350868	4.07e-582	1.97e-146	0.159
RAF	3	-1.4044916482153412260350868	2.36e-1116	2.62e-80	0.107
TSI	2	-1.4044916482153412260350868	2.08e-1755	3.10e-110	0.075
$f_8(x)$, $x_0 = 0.25$					
GE1	div	-	-	-	-
GE2	3	1.0000000000000000000000000000	1.73e-791	2.52e-57	0.025
THU	6	0.9999999999999999999999999999	1.17e-680	1.03e-170	0.073
RAF	3	0.9999999999999999999999999999	1.68e-1248	9.73e-90	0.028
TSI	3	0.9999999999999999999999999999	6.71e-1409	1.33e-88	0.017

5 Conclusion

In this work, we have proposed sixteenth order iterative methods. The convergence orders of the suggested methods were proved and the efficiency was also calculated. With the help of some test problems, a comparison of the obtained results with the existing methods such as the Geum and Kim [2, 3], Thukral [13] and Rafiullah and Jabeen [11] was also given and it was observed that the new methods are more efficient than the existing methods.

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