A study of generalization of bi-quasi-interior ideal of ordered semigroup.

Wichayaporn Jantanan

E-mail: wichayaporn.jan@bru.ac.th

Sruangsuda Kongkratok

E-mail: suangsuda71343@gmail.com

Atirach Rodcha

E-mail: atirach.rodcha@gmail.com

Department of mathematics/Faculty of science/Buriram Rajabhat University

ABSTRACT

In this paper, as a further generalization of ideals, we introduce the notion of bi-quasi-interior ideals as a generalization of bi-ideal, quasi ideal, interior ideal, bi-interior ideal and bi-quasi ideal of ordered semigroup and study the properties of bi-quasi-interior ideals of ordered semigroup.

KEYWORDS: bi-ideal, quasi-ideal, interior ideal, bi-interior ideal, bi-quasi ideal, bi-quasi-interior ideal, simple ordered semigroup and regular ordered semigroup.

1. Introduction

The study of ordered semigroup began about 1950 by several authors, the theory of different types of ideals in semigroup and in ordered semigroup was studies by several researches such as: In The notion of ordered semigroup was introduced by G. Brikhoff [1]. In 1998, N. Kehayopulu [5] gave some characterization of quasi-ideals and bi-ideals in completely regular ordered semigroup. In 1999 [4] the concept of interior ideals was introduced by N. Kehayopulu for ordered semigroups. In 2002, N. Kehayopulu, J. S. Ponizovskii and M. Tsingelis [3] studies bi-ideals in ordered semigroup and ordered group and introduced the notion of regular ordered semigroups and simple ordered semigroups. After, N. Kehayopulu [6] studies ideals in ordered semigroups and decomposition theorem of the left regular ordered semigroup into left simple ordered. In 2008, Y. Yin and H. Li [10] introduced and characterized the concept of regular and intra-regular semigroup. In 2015, N. Kehayopulu [7] introduced the notion of quasi-ideals of ordered semigroups. The concept of ordered (m,n) quasi ideals was introduced by Suthin and Aiyared Iampan [9]. Ronnason Chinram and Winida Yonthanthum [8] characterized the regularity-preserving element of regular ordered semigroup.In 2018, M. Murali Krishna Rao [2] studies ideals in semigroup and introduced the notion of bi-quasi-interior ideal as a generalization of quasi-ideal, bi-ideal and interior ideal of semigroup and studies the properties of quasi-ideal, bi-ideal and interior ideal of semigroup, simple semigroup and regular semigroup.

In this paper, as a further generalization of ideals, we introduce the notion bi-quasi-interior ideal as a generalization of bi-ideal, quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideal of ordered semigroup and study the properties of bi-quasi-interior ideal of ordered semigroup and some characterizations of bi-quasi-interior ideal of ordered semigroup, regular ordered semigroup and simple ordered semigroup.

2. Preliminaries and basic definitions

In this section we will recall some of the fundamental concept and definition, which are necessary for this paper.

Definition 1 [1] A semigroup $(S,\cdot)$ together with a partial order $\leq$ (on S) this is compatible with semigroup operation, meaning that for $x,y,z \in S$,

$$x \leq y \Rightarrow zx \leq zy \text{ and } xz \leq yz,$$

is called an ordered semigroup.

Let $(S,\cdot,\leq)$ be an ordered semigroup. If A and B are non-empty subsets of S, we let

$$AB = \{xy \in S \mid x \in A, y \in B\},$$

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Definition 2 [8] Let $(S,\cdot,\leq)$ be an ordered semigroup. A non-empty subset A of S is called a *subsemigroup* of S if $ab \in A$ for all $a,b \in A$. An element a of S is called an *idempotent* if $a \leq a^2$. An element $b \in S$ is called an *inverse* of a if $a \leq aba$ and $b \leq bab$. An element e of S is called an *identity* element of S if $ex = x = xe$ for any $x \in S$.

Definition 3 [6] Let $(S,\cdot,\leq)$ be an ordered semigroup. A non-empty subset A of S is called a left (resp,right) ideal of S if it satisfies the following condition:

(i) $SA \subseteq A$ (resp. $AS \subseteq A$). (ii) for $x \in A$ and $y \in S$, $y \leq x \Rightarrow y \in A$ or $(A] = A$.

Definition 4 [5] A non-empty subset B of an ordered semigroup $(S,\cdot,\leq)$ is called a bi-ideal of S if it satisfies the following condition:

(i) $BSB \subseteq B$. (ii) for $x \in B$ and $y \in S$, $y \leq x \Rightarrow y \in B$ or $(B] = B$.

Definition 5 [4] A subsemigroup B of an ordered semigroup $(S,\cdot,\leq)$ is called an interior ideal of S if it satisfies the following condition:

(i) $SBS \subseteq B$. (ii) for $x \in B$ and $y \in S$, $y \leq x \Rightarrow y \in B$ or $(B] = B$.

Definition 6 [7] A non-empty subset B of an ordered semigroup $(S,\cdot,\leq)$ is called a quasi-ideal of S if it satisfies the following condition:

(i) $(SB] \cap (BS] \subset B$. (ii) for $x \in B$ and $y \in S$, $y \leq x \Rightarrow y \in B$ or $(B] = B$.

Definition 7 [3] An ordered semigroup $(S,\cdot,\leq)$ is called a regular if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. Equivalent definition:

(i) $A \subseteq (ASA]$ $(\forall A \subseteq S)$. (ii) $a \in (aSa]$ $(\forall a \in S)$.

Definition 8 [3] An ordered semigroup $(S,\cdot,\leq)$ is said to be a left (respectively. right) simple ordered semigroup if S dose not contain proper left (respectively. right) ideal of S.

Definition 9 A non-empty subset B of an ordered semigroup S is said to be bi-interior ideal of S if it satisfies the following condition:

(i) $(SBS]\cap(BSB]\subseteq B$. (ii) for $a\in B$ and $b\in S, b\leq a\Rightarrow b\in B$ or $(B]=B$.

Definition 10 A subsemigroup A of an ordered semigroup $(S,\cdot,\leq)$ is called a left (respectively. right) bi-quasi ideal of S if it satisfies the following condition:

(i) $(SA]\cap(ASA]\subseteq A$ (respectively. $(AS]\cap(ASA]\subseteq A$);

(ii) for $a\in A$ and $b\in S, b\leq a\Rightarrow b\in A$ or $(A]=A$.

Definition 11 A subsemigroup A of an ordered semigroup $(S,\cdot,\leq)$ is called a bi-quasi ideal of S if A is a left bi-quasi ideal and a right bi-quasi ideal of S .

Lemma 1 [9] Let S be an ordered semigroup, A and B are subsets of S. Then the following statements hold.

(i) $A\subseteq(A]$. (ii) $((A]]=(A]$. (iii) If $A\subseteq B$, then $(A]\subseteq(B]$. (iv) $(A\cap B]\subseteq(A]\cap(B]$. (v) $(A\cup B]\subseteq(A]\cup(B]$. (vi) $(A](B]\subseteq(AB]$. (vii) $((A](B]]=(AB]$.

3. Bi-quasi-interior ideals of ordered semigroup

In this section, we introduce the notion of bi-quasi-interior ideal as a generalization of bi-ideal and quasi-ideal of ordered semigroup and study the properties of bi-quasi-interior ideals of ordered semigroup.

Definition 3.1 A non-empty subset B of an ordered semigroup M is said to be a bi-quasi-interior ideal of M if it satisfies the following condition:

(i) $(BMBMB]\subseteq B$. (ii) for $a\in B$ and $b\in M, b\leq a\Rightarrow b\in B$ or $(B]=B$.

Example 3.1. The ordered semigroup $M=\{a,b,c,d\}$ and $B=\{a\}$. Define operation $\cdot$ on M by the following table;

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define a relation $\leq$ on M by $\leq:\{(a,a),(a,b),(b,b),(c,c),(d,d)\}$. Then B is a bi-quasi-interior ideal of M.

Theorem 3.1 Every bi-ideal of ordered semigroup M is a bi-quasi-interior ideal of ordered semigroup M.

Theorem 3.2 Every interior ideal of ordered semigroup M is a bi-quasi-interior ideal of ordered semigroup M.

Theorem 3.3 Let M be an ordered semigroup. Every left ideal is a bi-quasi-interior ideal of M.

Corollary 3.1 Let M be an ordered semigroup. Every right ideal is a bi-quasi-interior ideal of M.

Corollary 3.2 Let M be an ordered semigroup. Every ideal is a bi-quasi-interior ideal of M.

Theorem 3.4 Let M be an ordered semigroup. Every quasi-ideal is a bi-quasi-interior ideal of M.

Theorem 3.5 Let M be an ordered semigroup. Intersection of a right ideal and a left ideal of M is a bi-quasi-interior ideal of M.

Theorem 3.6 Let M be an ordered semigroup. If L is a left ideal and is a right ideal of ordered semigroup M and $B=(RL]$ then B is a bi-quasi-interior ideal of M.

Theorem 3.7 Let M be an ordered semigroup. If B is a bi-quasi-interior ideal and T is a non-empty subset of M such that $(BT]$ is a subsemigroup of M then $(BT]$ is a bi-quasi-interior ideal of M.

Proof. Assume that B is a bi-quasi-interior ideal and T is a non-empty subset of M and $(BT]$ is a subsemigroup of M. Then $((BT]M(BT]M(BT]]\subseteq((BM]M(BM]M(BT]]\subseteq(BMBMBT]\subseteq(BT]$. Thus $((BT]M(BT]M(BT]]\subseteq(BT]$. Let $x\in(BT]$ and $y\in M$ be such that $y\leq x$. Since $x\in(BT]$, we have $x\leq z$ for some $z\in BT$. Then $y\leq x\leq z$, which implies that $y\in(BT]$. Hence $(BT]$ is a bi-quasi-interior ideal of M.

Theorem 3.8 The intersection of a bi-quasi-interior ideal B of ordered semigroup M and a right ideal A of M is always bi-quasi-interior ideal of M.

Proof. Assume that $C=B\cap A$, where B is a bi-quasi-interior ideal of M and A is a right ideal of M. Now, $(CMCMC]\subseteq(BMBMB]\subseteq B$ and $(CMCMC]\subseteq(AMAMA]\subseteq(AAA]\subseteq(A]=A$. It is clear that $(CMCMC]\subseteq B\cap A=C$. Let $x\in C$ and $y\in M$ such that $y\leq x$. Since $x\in B\cap A$, we have $y\in B$ and $y\in A$. The assumption applies $y\in C$. Hence the intersection of a bi-quasi-interior ideal B of ordered semigroup M and a right ideal A of M is always bi-quasi-interior ideal of M.

Theorem 3.9 Let A and C be bi-quasi-interior ideal of ordered semigroup M and $B=(AC]$. If $CC=C$ then B is bi-quasi-interior ideal of M.

Proof. Let A and C be bi-quasi-interior ideal of ordered semigroup M and $B=(AC]$. We have $BB=(AC](AC]\subseteq(ACAC]\subseteq(ACCAC]\subseteq(ACCCAC]\subseteq(ACMCMC]\subseteq(AC]=B$. Thus $B=(AC]$ is a subsemigroup of M. Now, $(BMBMB]=((AC]M(AC]M(AC]]\subseteq((ACMACMAC]]\subseteq(ACMACMAC]$ $\subseteq(AMAMAC]\subseteq(AC]=B$. Let $x\in(AC]$ and $y\in M$ be such that $y\leq x$. Since $x\in(AC]$, we have $x\leq z$ for some $z\in AC$. Then $y\leq x\leq z$, which implies that $y\in(AC]$. Hence B is a bi-quasi-interior ideal of M.

Corollary 3.3 Let A and C be a bi-quasi-interior ideal of ordered semigroup M and $B=(CA]$. If $CC=C$ then B is bi-quasi-interior ideal of M.

Theorem 3.10 Let A and C be subsemigroup of M and $B=(AC]$. If A is a left ideal of M then B is a bi-quasi-interior ideal of M.

Proof. Let A and C be subsemigroup of M and $B=(AC]$. Suppose that A is a left ideal of M. We see that $BB=(AC](AC]\subseteq(ACAC]\subseteq(AMAC]\subseteq(AAC]\subseteq(AC]=B$. Therefore $B=(AC]$ is a subsemigroup of M. Now, $(BMBMB]=((AC]M(AC]M(AC]]\subseteq((ACMACMAC]]\subseteq(ACMACMAC]$

$\subseteq (AC] = B.$ Let $x \in (AC]$ and $y \in M$ be such that $y \leq x.$ Since $x \in (AC],$ we have $x \leq z$ for some $z \in AC.$ Then $y \leq x \leq z,$ which implies that $y \in (AC].$ Hence B is a bi-quasi-interior ideal of $M.$

Corollary 3.4 Let A and C be a subsemigroup of M and $B = (AC].$ If A is a right ideal then B is a bi-quasi-interior ideal of $M.$

Theorem 3.11 B is a bi-quasi-interior ideal of ordered semigroup M if and only if B is a left ideal of some right ideal of an ordered semigroup $M.$

Proof. Assume that B is a bi-quasi-interior ideal of an ordered semigroup $M.$ Then $(BMBMB] \subseteq B.$
It follow that $(BMBM]M = (BMBM](M] \subseteq (BMBMM] \subseteq (BMBM].$ Let $x \in (BMBM]$ and $y \in M$ be such that $y \leq x.$ Since $x \in (BMBM],$ we have $x \leq z$ for some $z \in BMBM.$ Then $y \leq x \leq z,$ which implies that $y \in (BMBM].$ Therefore $(BMBM]$ is a right ideal of $M.$ And $(BMBM]B = (BMBM](B] \subseteq (BMBMB]$ $\subseteq (B] = B.$ Thus B is a left ideal of some right ideal of $M.$

Conversely, assume that B is a left ideal of a right ideal R of $M.$ We have $RM \subseteq R$ and $RB \subseteq B.$ Then $(BMBMB] \subseteq (RMMMB] \subseteq (RMB] \subseteq (RB] \subseteq (B] = B.$ That $(BMBMB] \subseteq B.$ Let $x \in B$ and $y \in M$ such that $y \leq x,$ then $y \in B.$ Hence B is a bi-quasi-interior ideal of $M.$

Corollary 3.5 B is a bi-quasi-interior ideal of an ordered semigroup M if and only if B is a right ideal of some left ideal of an ordered semigroup.

Theorem 3.12 If B is a bi-quasi-interior ideal of an ordered semigroup $M,$ T is a subsemigroup of M and $T \subseteq B$ then $(BT]$ is a bi-quasi-interior ideal of $M.$

Proof. Assume that B is a bi-quasi-interior ideal of an ordered semigroup $M,$ T is a subsemigroup of M and $T \subseteq B.$ Then, $(BT](BT] \subseteq (BTBT] \subseteq (BT].$ Thus $(BT]$ is a subsemigroup of $M.$ Now, $((BT]M(BT]M(BT]] \subseteq (BTMBTMBT] \subseteq (BMBMBT] \subseteq (BT].$ Let $x \in (BT]$ and $y \in M$ be such that $y \leq x.$ Since $x \in (BT],$ we have $x \leq z$ for some $z \in BT.$ Then $y \leq x \leq z,$ which implies that $y \in (BT].$ Hence $(BT]$ is a bi-quasi-interior ideal of $M.$

Theorem 3.13 Let B be a bi-ideal of an ordered semigroup M and I be an interior ideal of $M.$ Then $(B \cap I)$ is a bi-quasi-interior ideal of $M.$

Proof. Let M is an ordered semigroup. Suppose that B is a bi-ideal of M and I is an interior ideal of $M.$ Then $(B \cap I)(B \cap I) \subseteq BB \subseteq B$ and $(B \cap I)(B \cap I) \subseteq II \subseteq I.$ Therefore $(B \cap I)(B \cap I) \subseteq B \cap I.$ Hence $(B \cap I)$ is a subsemigroup of $M.$ Now, $((B \cap I)M(B \cap I)M(B \cap I)] \subseteq (BMBMB] \subseteq (BMMMB]$ $\subseteq (BMB] \subseteq (B] = B$ and $((B \cap I)M(B \cap I)M(B \cap I)] \subseteq (IMIMI] \subseteq (IMMMI] \subseteq (IMI] \subseteq (I] = I.$ Thus $((B \cap I)M(B \cap I)M(B \cap I)] \subseteq (B \cap I).$ Let $x \in B \cap I$ and $y \in M$ such that $y \leq x,$ then $y \in B \cap I.$ Hence $B \cap I$ is a bi-quasi-interior ideal of $M.$

Theorem 3.14 Let M be an ordered semigroup. If $M = (Ma]$ for all $a \in M,$ then every bi-quasi-interior ideal of M is a quasi ideal of $M.$

Proof. Let B is a bi-quasi-interior ideal of ordered semigroup M and $a \in B.$ Then $(BMBMB] \subseteq B.$

We have $M = (Ma] \subseteq (MB], a \in B.$ It is clear that $M \subseteq (MB] \subseteq (MM] \subseteq (M] = M.$ Hence $(MB] = M.$ We have $(BM] = (BMB] = (BMBM] = (BMBBB] \subseteq (BMBMB] \subseteq B.$ Therefore $(MB] \cap (BM] \subseteq M \cap B = B.$ Clearly, if $x \in B$ and $y \in M$ such that $y \le x,$ then $y \in B.$ Hence B is a bi-quasi-interior ideal of $M.$

Theorem 3.15 The intersection of bi-quasi-interior ideal $B_\lambda \mid \lambda \in A$ of an ordered semigroup M is a bi-quasi-interior ideal of $M.$

Proof. Let $B = \bigcap_{\lambda \in A} B_\lambda.$ Then B is a subsemigroup of $M.$ Since B_λ is a bi-quasi-interior ideal of $M,$ we get $(B_\lambda M B_\lambda M B_\lambda] \subseteq B_\lambda$ for all $\lambda \in A.$ Hence $((\bigcap B_\lambda)M(\bigcap B_\lambda)M(\bigcap B_\lambda)] \subseteq \bigcap B_\lambda = B.$ Let $x \in B$ and $y \in M$ such that $y \le x,$ then $y \in B.$ Therefore B is a bi-quasi-interior ideal of $M.$

Theorem 3.16 Let B be an interior ideal of ordered semigroup $M, e \in B$ and e be an idempotent. Then $(eB]$ is a bi-quasi-interior ideal of $M.$

Proof. Let B be an interior ideal of ordered semigroup $M,$ obviously $(eM]$ is a subsemigroup of $M.$ By considering $(eM]M(eM] \subseteq (eM](M](eM] \subseteq (eMMeM] \subseteq (eM].$ Let $x \in (eM]$ and $y \in M$ be such that $y \le x.$ Since $x \in (eM],$ we have $x \le z$ for some $z \in eM.$ Then $y \le x \le z,$ which implies that $y \in (eM].$ Hence $(eM]$ is a bi-ideal of $M.$ Suppose that $x \in B \cap (eM],$ then $x \in B$ and $x \in (eM].$ Since $x \in (eM],$ we have $x \le ey,$ for some $y \in M.$ Then $x \le ey \le eey \le eeey \le eeeey \in eBBBM \subseteq eBMBM \subseteq eBB \subseteq eB.$ Hence $x \in (eB].$ Therefore $B \cap (eM] \subseteq eB].$ Since $(eB] \subseteq (B] = B$ and $(eB] \subseteq (eM],$ we have $(eB] \subseteq B \cap (eM].$ This show that $B \cap (eM] = (eB].$ By theorem 3.13 we get that $(eB]$ is a bi-quasi-interior ideal of $M.$

Corollary 3.6 Let M be an ordered semigroup and e be an idempotent. Then $(eM]$ and $(Me]$ are bi-quasi-interior ideal of $M.$

Theorem 3.17 If B be a left bi-quasi ideal of ordered semigroup $M,$ then B is a bi-quasi-interior ideal of $M.$

Proof. Assume that B is a left bi-quasi ideal of ordered semigroup $M.$ Now, $(BMBMB] \subseteq (MB]$ and $(BMBMB] \subseteq (BMB].$ We have $(BMBMB] \subseteq (MB] \cap (BMB] \subseteq B.$ Clearly, if $x \in B$ and $y \in M$ such that $y \le x$, then $y \in B.$ Hence B is a bi-quasi-interior ideal of $M.$

Corollary 3.7 If B be a right bi-quasi ideal of ordered semigroup $M,$ then B is a bi-quasi-interior ideal of $M.$

Corollary 3.8 If B be a bi-quasi ideal of ordered semigroup $M,$ then B is a bi-quasi-interior ideal of $M.$

Theorem 3.18 If B be a bi-interior ideal of ordered semigroup $M,$ then B is a bi-quasi-interior ideal of $M.$

Proof. Suppose that B is a bi-interior ideal of ordered semigroup $M.$ We see that $(BMBMB] \subseteq (MBM]$ and $(BMBMB] \subseteq (BMB],$ we obtain $(BMBMB] \subseteq (MBM] \cap (BMB] \subseteq B.$ Clearly, if $x \in B$ and $y \in M$ such that $y \le x,$ then $y \in B.$ This proves that B is a bi-quasi-interior ideal of $M.$

We introduce the notion of bi-quasi-interior simple ordered semigroup and characterize the bi-quasi-interior simple ordered semigroup using bi-quasi-interior ideals of ordered semigroup and study the properties of minimal bi-quasi-interior ideal of ordered semigroup.

Definition 3.2 An ordered semigroup M is said to be bi-quasi-interior simple ordered semigroup if M has no bi-quasi-interior ideals other than M itself.

Theorem 3.19 Let M be a simple ordered semigroup. Every bi-quasi-interior ideal is a bi-ideal of M.

Proof. Let B be a bi-quasi-interior ideal of a simple ordered semigroup M. Then $(MBM]M \subseteq (MBMM]$ $\subseteq (MBM]$ and $M(MBM] \subseteq (MMBM] \subseteq (MBM]$. Let $x \in (MBM]$ and $y \in M$ be such that $y \leq x$. Since $x \in (MBM]$, we have $x \leq z$ for some $z \in MBM$. Then $y \leq x \leq z$, which implies that $y \in (MBM]$. Therefore $(MBM]$ is an ideal of M. Since M is a simple ordered semigroup, We have $(MBM] = M$. Now we prove that B is a bi-ideal of M. Consider $BMB \subseteq (BMB] = (B(MBM]B] \subseteq (BMBMB]$ $\subseteq (B(MBM]B(MBM]B] \subseteq (BMBMBMBMB] \subseteq (BMBM(BMBMB]] \subseteq (BMBMB] \subseteq B$. Clearly, if $x \in B$ and $y \in M$ such that $y \leq x$, then $y \in B$. It implies that B is a bi ideal of M. Hence the theorem is prove.

Theorem 3.20 Let M be an ordered semigroup. Then M is a bi-quasi-interior simple ordered semigroup if and only if $(a)_{bqi} = M$, for all $a \in M$, where $(a)_{bqi}$ is bi-quasi-interior ideal generated by a.

Proof. Let M be an ordered semigroup. Suppose that $(a)_{bqi}$ is bi-quasi-interior ideal generated by a. and M is a bi-quasi-interior simple ordered semigroup.Then $(a)_{bqi} = M$, for all $a \in M$.

Conversely, suppose B is a bi-quasi-interior of ordered semigroup M and $(a)_{bqi} = M$, for all $a \in M$. Let $b \in B$. then $(b)_{bqi} \subseteq B$ implies $M = (b)_{bqi} \subseteq B \subseteq M$. Therefore M is a bi-quasi-interior simple ordered semigroup.

Theorem 3.21 Let M be an ordered semigroup. M is a bi-quasi-interior simple ordered semigroup if and only if $\langle a \rangle = M$, for all $a \in M$ and where $\langle a \rangle$ is the smallest bi-quasi-interior ideal generated by a.

Proof. Let M be an ordered semigroup. Suppose that M is a bi-quasi-interior simple ordered semigroup, $a \in M$ and $B = (Ma]$. Then B is a left ideal of M. By theorem 3.3 B a bi-quasi-interior ideal of M. Therefore $B = M$. Hence $(Ma] = M$, for all $a \in M$.

$$(Ma] \subseteq \langle a \rangle \subseteq M,$$
$$M \subseteq \langle a \rangle \subseteq M,$$
$$M = \langle a \rangle.$$

Suppose $\langle a \rangle$ is the smallest bi-quasi-interior ideal of M generated by a, $\langle a \rangle = M$, A is a bi-quasi-interior ideal and $a \in A$. Then,

$$\langle a \rangle \subseteq A \subseteq M,$$
$$M \subseteq A \subseteq M.$$

Hence $A = M$. Hence M is a bi-quasi-interior simple ordered semigroup.

Theorem 3.22 Let M be an ordered semigroup. Then M is a bi-quasi-interior simple ordered semigroup if and only if $(aMaMa] = M$, for all $a \in M$.

Proof. Suppose that M is a bi-quasi-interior simple ordered semigroup and $a \in M$. Now, $((aMaMa]M(aMaMa]M(aMaMa]] \subseteq (aMaMaMaMaMaMaMaM] \subseteq (aMaMa]$. Let $x \in (aMaMa]$ and $y \in M$ be such that $y \leq x$. Since $x \in (aMaMa]$, we have $x \leq z$ for some $z \in aMaMa$. Then $y \leq x \leq z$, which implies that $y \in (aMaMa]$. Hence $(aMaMa]$ is a bi-quasi-interior ideal of M.

Conversely, suppose $(aMaMa]=M$, for all $a \in M$. Let B be a bi-quasi-interior ideal of ordered semigroup M and $a \in B$. Then, $M=(aMaMa] \subseteq (BMBMB] \subseteq B$. So $M=B$, which implies that M is a bi-quasi-interior simple ordered semigroup.

Definition 3.3 An ordered semigroup is a left(right) simple ordered semigroup if M has no proper left(right) ideal of M.

Theorem 3.23 If ordered semigroup M is left simple ordered semigroup then every bi-quasi-interior ideal of M is a right ideal of M.

Proof. Let B be a bi-quasi-interior ideal of left simple ordered semigroup. It is easy to see that $(MB]$ is a left ideal of M and $(MB] \subseteq M$. Let $x \in (MB]$ and $y \in M$ be such that $y \leq x$. Since $x \in (MB]$, we have $x \leq z$ for some $z \in MB$. Then $y \leq x \leq z$, which implies that $y \in (MB]$. Therefore $(MB]=M$. Since BM $=B(MB]=B((MB]B] \subseteq B((MB]M]=B((MB](MB]] \subseteq (BMBMB] \subseteq B$. So $BM \subseteq B$. Hence every bi-quasi-interior ideal is a right ideal of M.

Corollary 3.9 If ordered semigroup M is right simple ordered semigroup then every bi-quasi-interior ideal of M is a left ideal of M.

Corollary 3.10 Every bi-quasi-interior ideal of left and right simple ordered semigroup M is an ideal of M.

Theorem 3.24 Let M be an ordered semigroup and B be a bi-quasi-interior ideal of M. Then B is minimal bi-quasi-interior ideal of M if and only if B is a bi-quasi-interior simple ordered semigroup.

Proof. Let B be a minimal bi-quasi-interior ideal of ordered semigroup M and C be bi-quasi-interior ideal of B. Then $(CBCBC] \subseteq C$. Now, $((CBCBC]M(CBCBC]M(CBCBC]] \subseteq (CBCBCMCBCBCMCBCBC]$ $\subseteq (C(BMBMB]C(BMBMB]C] \subseteq ((C](BMBMB](C](BMBMB](C]] \subseteq (CBCBC] \subseteq C$. Clearly, if $x \in (CBCBC]$ and $y \in M$ be such that $y \leq x$. Since $x \in (CBCBC]$, we have $x \leq z$ for some $z \in CBCBC$. Then $y \leq x \leq z$, which implies that $y \in (CBCBC]$. This show that $(CBCBC]$ is a bi-quasi-interior ideal of M. Since C is a bi-quasi-interior ideal of B then $B=(CBCBC] \subseteq C$. So $B=C$.

Conversely, assume B is a bi-quasi-interior simple ordered semigroup of M. Let C be a bi-quasi-interior ideal of M and $C \subseteq B$. Then we have $C=(CBCBC] \subseteq (CMCMC] \subseteq (BMBMB] \subseteq B$. So $B=C$. Since B is a bi-quasi-interior simple ordered semigroup. Hence B is a minimal bi-quasi-interior ideal of M.

Theorem 3.25 Let M be an ordered semigroup and $B=(RL]$, where L and R are minimal left ideal and right ideal of M respectively. Then B is a minimal bi-quasi-interior ideal of M.

Proof. Let M be an ordered semigroup and B is a bi-quasi-interior ideal of M. Then $(BMBMB]$ $\subseteq ((RL]M(RL]M(RL]] \subseteq (RMMRLMML] \subseteq (RMRLML] \subseteq (RRLL] \subseteq (RL]=(B]$. Let $x \in (RL]$ and $y \in M$ be such that $y \leq x$. Since $x \in (RL]$, we have $x \leq z$ for some $z \in RL$. Then $y \leq x \leq z$, which implies that $y \in (RL]$. Hence $B=(RL]$ is bi-quasi-interior ideal of M. Let A be a bi-quasi-interior ideal of M. Such that $A \subseteq B$. It is obvious to see that $(MA]$ is a left ideal and $(AM]$ is a right of M. Then $(MA] \subseteq (MB] \subseteq (M(RL]] \subseteq (MRL] \subseteq (ML] \subseteq L$, since L is a left ideal of M. Similarly, we can prove

$(AM] \subseteq (BM] \subseteq ((RL]M] \subseteq (RLM] \subseteq (RM] \subseteq R,$ since R is a right ideal of $M.$ Using the minimality of L and $R,$ we get $(MA] = L$ and $(AM] = R.$ Now, $B = (RL] = ((AM](MA]] \subseteq ((AMMA]] \subseteq ((BMMB]]$ $\subseteq ((RL]MM(RL]] \subseteq (RLMMRL] \subseteq ((AM](MA]M(MA]] \subseteq (AMMAMMA] \subseteq (AMAMA] \subseteq A.$ So $A = B.$ Hence B is a minimal bi-quasi-interior ideal of $M.$

Theorem 3.26 Let M be a regular ordered semigroup. Then every interior ideal of M is an ideal of $M.$

Proof. Let B be an interior ideal of M by theorem 3.2, we have B is a bi-quasi interior ideal of $M.$ Then $(BMBMB] \subseteq B.$ Since M is regular, we have $BM \subseteq (BMB]M \subseteq (BMBM] \subseteq (BM(BMB]M]$ $\subseteq (BMBMBM] \subseteq (BMBM(BMB]M] \subseteq (BMBMBMBM] \subseteq (BMBMB] \subseteq B$ and $MB \subseteq M(BMB]$ $\subseteq (MBMB] \subseteq (MBMBMB] \subseteq (MBMBMBMB] \subseteq (BMBMB] \subseteq B.$ Clearly, if $x \in B$ and $y \in M$ such that $y \leq x,$ then $y \in B.$ Hence the theorem is proved.

Theorem 3.27 [10] M is regular ordered semigroup if and only if $AB = A \cap B$ for any right ideal A and left ideal B of ordered semigroup $M.$

Theorem 3.28 Let M be a regular ordered semigroup. Then B is bi-quasi-interior ideal of M if and only if $(BMBMB] = B,$ for all bi-quasi-interior ideal of $M.$

Proof. Suppose that M is a regular ordered semigroup, B is bi-quasi-interior ideal of M and $x \in B.$ Then $(BMBMB] \subseteq B$ and there exists $y \in M$ such that $x \leq xyx \leq xyxyx \in BMBMB.$ This show that $x \in (BMBMB].$ Hence $(BMBMB] = B.$

Conversely, suppose $(BMBMB] = B,$ for all B is bi-quasi-interior ideal of $M.$ Let $B = R \cap L,$ where R is a right ideal and L is a left ideal of $M.$ Therefore $R \cap L = ((R \cap L)M(R \cap L)M(R \cap L)] \subseteq (RMLML]$ $\subseteq (RL] \subseteq R \cap L,$ since $(RL] \subseteq (L] = L$ and $(RL] \subseteq (R] = R.$ This show that $R \cap L = (RL].$ Hence by theorem 3.27, M is a regular ordered semigroup.

Theorem 3.29 Let B be a subsemigroup of regular ordered semigroup M then B can be represented as $B = (RL],$ where R is a right ideal and L is a left ideal of M if and only if B is a bi-quasi-interior ideal of $M.$

Proof. Suppose that $B = (RL],$ where R is a right ideal and L is a left ideal of $M.$ The following holds $(BMBMB] = ((RL]M(RL]M(RL]] \subseteq (RLMRLMRL] \subseteq (RL] = B.$ Let $x \in (RL]$ and $y \in M$ be such that $y \leq x.$ Since $x \in (RL],$ we have $x \leq z$ for some $z \in RL.$ Then $y \leq x \leq z,$ which implies that $y \in (RL].$ Thus B is a bi-quasi-interior ideal of ordered semigroup $M.$ Conversely, suppose that B is a bi-quasi-interior ideal of regular ordered $M.$ By theorem 3.28, we have $(BMBMB] = B.$ Setting $R = (BM]$ and $L = (MB].$ Then $R = (BM]$ is a right ideal of M and $L = (MB]$ is a left ideal of $M.$ The following hold $(BM] \cap (MB] = (BM](MB] \subseteq (BMMB] \subseteq (BMBMB] = B.$ Hence $R \cap L = (BM] \cap (MB] \subseteq B.$ Since $B \subseteq (BM] = R$ and $B \subseteq (MB] = L,$ we have $B \subseteq R \cap L.$ It is clear that $B = R \cap L = (RL].$ Thus B can represented as $(RL],$ where R is a right ideal and L is a left ideal of $M.$ Hence the theorem is proved.

Theorem 3.30 M is regular ordered semigroup if and only if $B \cap I \cap L \subseteq (BIL],$ for any bi-quasi-interior ideal $B,$ ideal $I,$ and left ideal of $M.$

Proof. Suppose that M be a regular ordered semigroup, B, I and L are bi-quasi-interior ideal, ideal and left ideal of M respectively. Let $a \in B \cap I \cap L$. Since M is a regular, we have $a \in (aMa]$. So $a \leq aya$ for some $y \in M$. Then $a \leq aya \leq ayaya \in BMIML \subseteq BIL$. Thus $a \in (BIL]$. It is clear that $B \cap I \cap L \subseteq (BIL]$. Conversely, suppose that $B \cap I \cap L \subseteq (BIL]$, for any bi-quasi-interior ideal B, ideal I and left ideal L of M. Let R be a right ideal and L be a left ideal of M. Then by assumption $R \cap L = R \cap M \cap L$ $\subseteq (RML] \subseteq (RL]$. We have $(RL] \subseteq (RM] \subseteq (R] = R$ and $(RL] \subseteq (ML] \subseteq (L] = L$. Therefore $(RL] \subseteq R \cap L$.

4. Conclusion

Here we introduced the notion of bi-quasi-interior ideal of ordered semigroup as a generalization of bi-ideal, quasi ideal, interior ideal, bi-interior ideal and bi-quasi ideal of ordered semigroup and studied some of their properties. We introduced the notion of bi-quasi-interior simple ordered semigroup and characterized bi-quasi-interior simple ordered semigroup, regular ordered semigroup using bi-quasi-interior ideals of ordered semigroup. We proved every bi-quasi ideal of ordered semigroup and bi-interior ideal of ordered semigroup are bi-quasi-interior ideal. In continuity of this paper, we study bi-quasi-interior ideals of gamma ordered semigroup, ordered semihypergroup.

5. References

[1] G. Birkhoff. (1967). Lattice theory. New York: Amer. Math. Soc. Coll. Publ., Vol. 25, Providence.

[2] M. M. K. Rao. (2018). A study of a generalization of bi-ideal quasi ideal an interior ideal of semigroup. Mathematica Moravica, Vol. 22, p. 103-115.

[3] N. Kehayopulu, J. S. Ponizovskii, and M. Tsingelis. (2002). Bi-ideals in ordered semigroou and ordered groups. J. Math. Sci., New York. Vol. 112, No. 4.

[4] N. Kehayopulu. (1999). Note on interior ideals, ideal elements in ordered semigroups. Scientiae Mathematicae, Vol. 2, No. 3, p. 407-409.

[5] N. Kehayopulu. (1998). On completely regular ordered semigroup. Sci Math, 1, p. 27-32.

[6] N. Kehayopulu and M. Tsingelis. (2002). On left regular ordered semigroups. Southeast Asian Bull. Math, 25, p. 609-615.

[7] N. Kehayopulu. (2015). Remark on quasi-ideal of ordered semigroup. PU. M. A. Vol. 25, no. 2, p. 144-150.

[8] R. Chinram and W. Yonthanthum. (2019). On the regularily-preserving element in regular ordered semigroup. International Journal of Mathematics and Computer Science, No. 3, p. 729-736.

[9] Suthin Thongrak and Aiyared Iampan. (20018). characterizations of ordered semigroups by the properties of their ordered (m,n) Quasi ideals. Palestine Journal of Mathematics, Vol. 7 (1).

[10] Y. Yin and H. Li. (2008). The characterization of regular and intra-regular Semigroups. Ital J. Pure Appl. Math, 24, p. 41-52.