On 0-Minimal bi-Hyperideals of Semihypergroups with Zero

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Abstract

In this paper, we study minimal bi-hyperideals in semihypergroups. The notions of minimal bi-hyperideals in semihypergroups are introduced and described. The results obtained extend the results on semigroup.

Keywords : bi-hyperideal, semihypergroup

1.Introduction

Algebraic hyperstructures are a generalization of a classical algebraic structure and they were introduced by F. Marty[1]. In this paper, we consider concept of bi-hyperideal in semihypergroups. The notions of bi-hyperideal was introduced by S. Lekkoksung[2]. The auther also introduced the notions of bi-hyperideal in semihypergroups. We extend the results in [3] and [4] to semihypergroups. The rest of this section let us recall some terminologies and results used throughout the paper.

A hyperoperation on a nonempty set H is a map $\circ : H \times H \to P^*(H)$ where $P^*(H)$ is the family of nonempty subset of H. if A and B are nonempty subsets of H and $x \in H$, then we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b; \; x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A semihypergroup is a system $(H, \circ)$ where H is noempty subset, $\circ$ is a hyperoperation on H and $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$. An element e of a semihypergroup H is called an identity of $(H, \circ)$ if $x \in (x \circ e) \cap (e \circ x)$ for all $x \in H$ and it is called a scalar identity of $(H, \circ)$ if $(x \circ e) \cap (e \circ x) = \{x\}$ for all $x \in H$. A semihypergroup H with an element 0 such that $0 \circ x = x \circ 0 = \{0\}$ for all x in H, then 0 is said to be a zero element of H and H is called a semihypergroup with zero.

A nonempty subset A of a semihypergroup H is called a subsemihypergroup of H if $A \circ A \subseteq A$ and if $H \circ A \subseteq A (A \circ H \subseteq A)$, then A is called a left hyperideal (right hyperideal) of H. Moreover, if A is a left and a right hyperideal of H, then it is called a hyperideal of H.

Definition 1.1 A subsemihypergroup A of H is called a bi-hyperideal of H if $A \circ H \circ A \subseteq A$.

Definition 1.2 A bi-hyperideal A of a semihypergroup H with zero is called degernerate if $A = \{0, a\}$ with $a \circ H^1 \circ a = \{0\}$ such that 1 is a scalar identity.

Example 1.3 Let $H = \{0, 1, a, b\}$. Define a hyperoperation $\circ$ on H by

$\circ$	0	1	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{0\}$	$\{a\}$	$\{0\}$	$\{0, a\}$
b	$\{0\}$	$\{b\}$	$\{0, a\}$	H

Then $(H, \circ)$ is a semihypergroup. Let $A = \{0, a\}$, we have A is a subsemihypergroup of H. Since $\{0, a\} \circ H \circ \{0, a\} = \{0, a\}$, then A is a bi-hyperideal of H and it easy to see that $a \circ H \circ a = \{0\}$. Therefore A is a degenerate.

Definition 1.4 A left hyperideal (right hyperideal) A of a semihypergroup H with zero will be said to be 0-minimal left-hyperideal (right hyperideal) of H if $A \neq \{0\}$ and $\{0\}$ is the only left hyperideal (right hyperideal) of H properly contained in A.

In the above definition, if A is a 0-minimal left-hyperideal and 0-minimal right hyperideal of H, then A is a 0-minimal hyperideal of H.

Definition 1.5 if $A \neq \{0\}$ and $\{0\}$ is the only bi-hyperideal of H properly contained in A, then a bi-hyperideal A of a semihypergroup H with zero will be said to be 0-minimal bi-hyperideal of H.

2.Research Results

In this section, we study degenerate 0-minimal bi-hyperideal, non-degenerate 0-minimal bi-hyperideal and 0-minimal bi-hyperideal.

Lemma 2.1 Let H be a semihypergroup with zero and A be a 0-minimal bi-hyperideal of H. Then either $a \circ H \circ a = A$ for every a in $A \setminus \{0\}$ or A is degenerate.
Proof. Let $a \in A \setminus \{0\}$. Then $a \circ H \circ a \subseteq A \circ H \circ A \subseteq A$ and

$$\begin{aligned}(a \circ H \circ a) \circ H \circ (a \circ H \circ a) &= a \circ (H \circ a \circ H \circ a \circ H) \circ a \\ &\subseteq a \circ H \circ a\end{aligned}$$

so $a \circ H \circ a$ is a bi-hyperideal of H contained in A. Since A is a 0-minimal bi-hyperideal of H so $a \circ H \circ a = \{0\}$ or $a \circ H \circ a = A$. Let $a \circ H \circ a = \{0\}$. If $a \circ a = \{0\}$, we have A is a degenerate. If $a \circ a = \{a\}$ so $(a \circ a) \circ a = \{a\} \circ a = a \circ a = \{a\}$. This is impossible because $a \in a \circ a \circ a \subseteq a \circ H \circ a$. If $a \circ a = \{0, a\}$ then $(a \circ a) \circ a = \{0, a\} \circ a = 0 \circ a \cup a \circ a = \{0, a\}$. This is impossible, since $a \in a \circ a \circ a \subseteq a \circ H \circ a$. If there exist $x \in a \circ a$ such that $x \notin \{0, a\}$. Since $x \in a \circ a \subseteq A$, so $\{0, x\} \subset A$. It easy to see that

$$\begin{aligned}\{0, x\} \circ H \circ \{0, x\} &= 0 \circ H \circ 0 \cup 0 \circ H \circ x \cup x \circ H \circ 0 \cup x \circ H \circ x \\ &= \{0\} \subseteq \{0, x\}.\end{aligned}$$

Therefore $\{0, x\}$ is a bi-hyperideal of H. This is impossible because A is 0-minimal bi-hyperideal of H. We have $a \circ a = \{0\}$ and $A = \{0, a\}$ with $a \circ H^1 \circ a = \{0\}$ so that A is a degenerate bi-hyperideal.

It easy to see that the following statements holds:

Lemma 2.2 A subset A of a semihypergroup H with zero is a non-degenerate 0-minimal bi-hyperideal of H if and only if $A = a \circ H \circ a$ for every a in $A \setminus \{0\}$

Theorem 2.3 Let H be a semihypergroup with zero and A be a 0-minimal bi-hyperideal of H. Then either $A^2 = \{0\}$ or $x \circ A \circ x = A$ for every $x \in A$

Proof. Assume $A^2 \neq \{0\}$. Let $x \in A \setminus \{0\}$. By Lemma 2.1, $x \circ H \circ x = A$. It easy to see that

$$\begin{aligned}(x \circ A \circ x) \circ H \circ (x \circ A \circ x) &= x \circ A \circ (x \circ H \circ x) \circ A \circ x \\ &= x \circ A \circ A \circ A \circ x \\ &\subseteq x \circ A \circ x.\end{aligned}$$

Therefore $x \circ A \circ x$ is a bi-hyperideal contained in A, it follows that either $x \circ A \circ x = \{0\}$ or $x \circ A \circ x = A$. Assume that $x \circ A \circ x = \{0\}$. Thus $\{0\} = x \circ A \circ x = x \circ (x \circ H \circ x) \circ x = x^2 \circ H \circ x^2$. According to Lemma 2.1, we have $x^2 = \{0\}$. Then $A^2 = (x \circ H \circ x) \circ (x \circ H \circ x) = x \circ H \circ x^2 \circ H \circ x = \{0\}$. This is imposible. Therefore $x \circ A \circ x = A$.

Lemma 2.4 Let H be a semihypergroup with zero and A be a 0-minimal right hyperideal of some 0-minimal left hyperideal of H. Then A is a 0-minimal bi-hyperideal of H.

Proof. **Case 1.** L is a degenerate 0-minimal left hyperideal of H. Then $L = \{0, l\}$ and $H \circ l = \{0\}$. Since A is 0-minimal right hyperideal of L so $A = L$. Then $A \circ H \circ A = A \circ H \circ L \subseteq A \circ L \subseteq A$. Therefore A is a degenerate 0-minimal bi-hyperideal of H. **Case 2.** L is a non-degenerate 0-minimal left hyperideal of H. If A is a degenerate 0-minimal right hyperideal of L. Then $A = \{0, a\}$ with $a \circ L = \{0\}$ and $A \circ H \circ A \subseteq A \circ H \circ L \subseteq A \circ L \subseteq A$. Hence A is a degenerate 0-minimal bi-hyperideal of L. If A is a non-degenerate 0-minimal right hyperideal of L. Hence $H \circ a = L$ and $a \circ L = A$ for every a in $A \setminus \{0\}$. Thus $a \circ H \circ a = a \circ L = A$ for every a in $A \setminus \{0\}$. By Lemma 2.2, we have A is a non-degenerate 0-minimal bi-hyperideal of H.

Lemma 2.5 Let H be a semihypergroup with zero and A be a non-degenerate 0-minimal bi-hyperideal of H and $a \in A \setminus \{0\}$. Then

(i) $A = a \circ H \circ a$

(ii) If L is a left hyperideal of H contained in $H \circ a$ then $L^2 = \{0\}$ or $L = H \circ a$

(iii) If R is a right hyperideal of $H \circ a$ contained in A then $R^2 = \{0\}$ or $R = A$

Proof. (i) By Lemma 2.2, $A = a \circ H \circ a$.

(ii) Let $L \subseteq H \circ a$ and $H \circ L \subseteq L$. Since $(a \circ L) \circ H \circ (a \circ L) \subseteq (a \circ L) \circ H \circ L \subseteq a \circ L \circ L \subseteq a \circ L$ so $a \circ L$ is a bi-hyperideal contained in A. It follows that $a \circ L = \{0\}$ or $a \circ L = A$. If $a \circ L = \{0\}$ then $L^2 \subseteq (H \circ a) \circ L = H \circ (a \circ L) = \{0\}$. If $a \circ L = A$. Since $b \circ L \subseteq H \circ L \subseteq L$ implies $A \subseteq L$. Thus $H \circ a \subseteq H \circ L \subseteq L$. Therefore $L = H \circ a$.

(iii) Let $R \subseteq A$ and $R \circ H \circ a \subseteq R$. Since $(R \circ H \circ a) \circ H \circ (R \circ H \circ a) \subseteq R \circ H \circ a$ so $R \circ H \circ a$ is a bi-hyperideal of H contained in A. It follows that $R \circ H \circ a = \{0\}$ or $R \circ H \circ a = A$. If $R \circ H \circ a = \{0\}$ then $R^2 \subseteq R \circ R \subseteq R \circ H \circ a = \{0\}$. If $R \circ H \circ a = A$ then $R \circ H \circ a \subseteq R$ implies $A \subseteq R$. Therefore $A = R$.

Lemma 2.6 Let H be a semihypergroup with zero and A be a 0-minimal bi-hyperideal of H such that $x \circ A \circ x = A$ for every $x \in A \setminus \{0\}$. Then A is a 0-minimal right hyperideal of $H \circ x$.

Proof. Since $A \circ H \circ x \subseteq A \circ H \circ A \subseteq A$. Then A is a right hyperideal of $H \circ x$. If $R \subseteq A$ and R is a right hyperideal of $H \circ x$. By Lemma 2.5 (iii) we have $R^2 = \{0\}$ or $R = A$. If $x \in R \setminus \{0\}$ then $A = x \circ A \circ x \subseteq R \circ H \circ x \subseteq R$. Therefore A is a 0-minimal right hyperideal of $H \circ x$.

The following theory follows from Lemma 2.4 and Lemma 2.6:

Theorem 2.7 Let H be a semihypergroup without zero and A be a nonempty subset of H. Then A is a minimal bi-hyperideal of H if and only if A is a minimal right hyperideal of some minimal left hyperideal of H.

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